3^{dP}

ex1) "تحليل حقيقي"

Prove

① $P(1)$ is true

L.H.S R.H.S

$P(2)$: $1+2 = 3$, $\frac{1}{\cancel{2}}(\cancel{2})(2+1) = 3$

suppose that $P(n)$ is true

$$1+2+3+\dots+n = \frac{1}{2}n(n+1)$$

② $1+2+3+\dots+n+(n+1) = \frac{1}{2}n(n+1)+\frac{2}{2}(n+1)$

$$= \frac{1}{2}(n+1)[n+2]$$

$$= \frac{1}{2}(n+1)[(n+1)+1]$$

$\therefore P_{n+1}$ is true

$\Rightarrow P_n$ is true for any $n \in \mathbb{N}$

4

$5^n - 4n - 1$

ex2) تقبل القسمة على 16

$P_n: 5^n - 4n - 1 = 16m$

① $P(1): 5 - 4 - 1 = 0$ is divisable by 16

$P(2) = 25 - 8 - 1 = 16$

② suppose that P_n is true at $K \in \mathbb{N}$

$\therefore 5^K - 4K - 1 = 16m, \quad m \in \mathbb{Z}$

③

$P_{K+1}: 5^{K+1} - 4(K+1) - 1$

$5 \cdot 5^K - 4K - 4 - 1 = 5 \cdot 5^K - 4K - 5$

$5(5^K - 4K - 1) + 16K$

$5^{K+1} - 20K - 5 + 16K + 4 - 1$

$5^K - 4$

★ $5^K - 4K - 1 = 16K$

$5^{K+1} - 4(K+1) - 1$

$5 \cdot 5^K - 4K - 5 \quad = 16m$

$= 5(5^K - 4K - 1) + 16K \quad = K$

$5 \cdot 5^K - 20K - 5 + 16K = 5 \cdot 5^K - 4K - 5$

$5(16m) + 16m \Rightarrow 16(5m + m)$

$6(16m) = 16(6m)$

$\therefore P_{K+1}$ is true and P_n is true for any $n \in \mathbb{N}$

$\leq K|\sin x| + |\sin x|$

$= (K+1)|\sin x|$

$\therefore |\sin(K+1)x| \leq (K+1)|\sin x|$

$\therefore P_{K+1}$ is true ~~for~~ $\Rightarrow P_n$ is true for any $n \in N$

(8)

" تحليل "

Algebric number : $\Rightarrow$ "العدد جبري"

A number r is said to be ~~algebram~~ algebric no if at saties the Poly nomial

n $a_n x^{n} + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 = 0$

~~Exs~~ $a_n \neq 0$, $a_i \in Z$, $i = 1, 2, 3, \dots, n \geq 1$

Ex : showthat (I) $\sqrt[5]{9}$, (II) $\frac{\sqrt[3]{7-\sqrt{3}}}{2}$, are algebric no .

Sol : Let $x = \sqrt[5]{9}$ ①

$x^5 = 9$

للتحقق

$x^5 - 9 = 0 \Rightarrow (\sqrt[5]{9})^5 - 9$

$= (9^{\frac{1}{5}})^5 - 9$

$= 0$

$\therefore \sqrt[5]{9}$ is algebric no .

ex) ④ $\Rightarrow$ $x = 4$

نعوض عن x بـ 4 $\Leftarrow$ $x - 4 = 0$

$4 - 4 = 0$

2.2 Rational zeros theorem ١ اسم النظرية

$a_n x^2 + C_{n-1} x^{n-1} + \dots + C_1 x + C_0 = 0$

نفس الشيء

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 = 0$

$$r = \frac{c \to a_0 \text{ تقسم}}{d \to a_n \text{ تقسم}}, \quad n \geq 1, \quad a_n \neq 0 \neq a_0$$

العوض عن x بقيمة r

$a_n\left(\frac{c}{d}\right)^n + a_{n-1}\left(\frac{c}{d}\right)^{n-1} + \dots + a_2\left(\frac{c}{d}\right)^2 +$

$a_1\left(\frac{c}{d}\right) + a_0 = 0$

d^n

$a_n\left(\frac{c^n}{d^n}\right) + a_{n-1}\left(\frac{c^{n-1}}{d^{n-1}}\right) + a_2 \frac{c^2}{d^2} + a_1 \frac{c}{d} + a_0 = 0$

$a_n c^n + \cancel{a_{n-1}} + a_{n-1} d c^{n-1} + a_2 d^{n-2} c^2 + a_1 d^{n-1} c$
$+ d^n a_0$

b

$\cancel{a_n c^n} =$

$a_n c^n = -d(a_{n-1} c^{n-1} \dots - a_2 d^{n-3} - a_1 d^{n-2}$
$- d^{n-1} a_0)$

$\therefore d \mid a_n c^n, \ \cancel{(c,d)} \ \gcd(c,d) = 1$

$\therefore d \mid a_n$

$x = ab$

$a_n c^n + a_{n-1} d c^{n-1} + \cdots + a_2 d^{n-2} c^2 + a_1 d^{n-1} c + d^n a_0 = 0$

$\underbrace{d^n a_0}_{x} = -\overset{\overset{a}{\uparrow}}{c}\underbrace{(a_n c^{n-1} + a_{n-1} c^{n-2} + \cdots + a_2 d^{n-2} c + a_1 d^{n-1})}_{b}$

$c \mid d^n a_0$

$\therefore c \mid a_0$

(17)

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

" تحليل حقيقي "

3.5 Theorem

(i) $|a| \geq 0$

(ii) $|ab| = |a||b|$, (iii) $|a+b| \leq |a| + |b|$

Proof:

(II) Case I: if $a \geq 0$, $b \geq 0$

$$|a||b| = ab = |ab|$$

Case 2: if $a \geq 0$, $b \leq 0$ $\rightarrow -(-ab)$

$$|a||b| = a(-b) = -ab = |ab|$$

Case 3: if $a \leq 0$, $b \geq 0$

Case 4: if $a \leq 0$, $b \leq 0$

$$|a||b| = (-a)(-b) = ab = |ab|$$

(III) $-|a| \leq a \leq |a|$

$+ \quad -|b| \leq b \leq |b|$

$$-(|a|+|b|) \leq a+b \leq |a|+|b|$$

$$|a|+|b| \geq -(a+b) \Rightarrow -(a+b) \leq |a|+|b|$$

$$(a+b) \leq |a|+|b|$$

$$|a+b| \leq |a|+|b|$$

Max

Def :

let If $S \subseteq R$, $S \neq 0$

① if $s_0 \in S$: if $s \leq s_0 , \forall s \in S$

then s_0 called Max of S . $Max(S) = s_0$

② if $s \geq s_0 , \forall s \in S$, then s_0 called minimam of S , $min(S) = s_0$

Ex)

Let $S = \{ \frac{1}{2} , \frac{1}{6} , \frac{1}{3} , 2 , \overset{2.7}{e} , \overset{3.14}{\pi} \}$

$T = \{ n \in N : 3 \leq n \leq 5 \}$

$R = \{ x \in Z : -3 \leq x < 0 \}$

$W = \{ r \in Q : -\sqrt{2} < r < \sqrt{5} \} \Rightarrow -2 < r \leq \sqrt{5}$

find Maximum and minimum for each given set

$Max(S) = \pi$ $\quad$ $Mim(S) = \frac{1}{6}$

$Max(T) = 5$ $\quad$ $Mim(T) = 3$

$Max(R)$ = Not exist $\quad$ $Min(R) = -3$

$Max(W)$ = Not exist $\quad$ $Min(W)$ = Not exist .

$Z = [-5, -2)$

$Max(Z)$ = Not exist , $min(Z) = -5$

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" تحليل حقيقي " " مراجعه "

$(a_n)_{n\in N} = \{a_1, a_2, \ldots\}$ sequence " متتاليه "

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ series

شروط كوشي :

$\exists N : n, m > N \in N \Rightarrow |a_n - a_m| < \varepsilon$

$$\begin{array}{ccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ \{2, & 1, & \frac{1}{2}, & \frac{1}{4}, & \frac{1}{8}, & \frac{1}{16}, & \frac{1}{32}\} \end{array}$$

يكون قيمه اقل من $\frac{1}{18}$

$\varepsilon = \frac{1}{18}$ $\quad n, m > a_6$

" متسلسلات الدوال " اختبار متسلسلات

covergent test for series:

(1) Compairson test: $\sum a_n$, $\sum b_n$

(1) $a_n < b_n$, $\sum b_n$ convergent $\Rightarrow \sum a_n$ convergent

(2) $a_n > b_n$, $\sum b_n$ divergent $\Rightarrow \sum a_n$ divergent

(2) Ratio test: $\frac{a_{n+1}}{a_n}$ $\quad \alpha = \frac{a_{n+1}}{a_n}$

(2) $\sum a_n$, $\alpha = \lim \sup |a_n|^{1/n} \Rightarrow \alpha = \lim \{\frac{a_{n+1}}{a_n}\}$

(1) $\sum a_n$ converges absolutly if $\alpha < 1$

(2) $\sum a_n$ diverges if $\alpha > 1$

(3) if $\alpha = 1$ the test is foiled

use Ratio :

Ex) $a_n = \frac{(\sqrt{n})^n}{n!}$, $\sum a_n$? هل $\sum a_n$ تقاربية او تباعدية

$1 + \frac{(\sqrt{2})^2}{2!} + \left\{\frac{(\sqrt[3]{3})}{3!}\right\}^3 + \dots$

$* (n+1)! = (n+1)n \dots$
$= (n+1)\, n!$

$a_{n+1} = \frac{(\sqrt{n+1})^{n+1}}{(n+1)!}$

$\frac{a_{n+1}}{a_n} = \frac{(\sqrt{n+1})^{n+1}}{(n+1)!} \cdot \frac{(n+1)!}{(\sqrt{n})^n}$

$= \frac{\sqrt{n+1}\,(\sqrt{n+1})^n}{(\sqrt{n})^n} \cdot \frac{n!}{(n+1)\cancel{!}}$

$= \sqrt{n+1}\left(\frac{\sqrt{n+1}}{\sqrt{n}}\right)^n \cdot \frac{1}{n+1}$

$\cancel{\alpha = \lim \frac{a_{n+1}}{a_n}}$

$\alpha = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \cdot \sqrt{\left(1+\frac{1}{n}\right)^{n}} \Rightarrow e$

$= 0 \cdot e = 0$

$\therefore \sum a_n$ is converges

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Root test

$\sum a_n$

$$\alpha = \lim_{n \to \infty} \sup |a_n|^{\frac{1}{n}} \begin{cases} < 1 & \text{convergence} \\ > 1 & \text{diverge} \end{cases}$$

Ex: consider $\sum \underbrace{\frac{n}{3^n}}_{a_n}$ $\boxed{\lim n^{\frac{1}{n}} = 1}$

$$\alpha = \lim \sup \left| \frac{n}{3^n} \right|^{\frac{1}{n}} = \lim \sup \left| \frac{n^{\frac{1}{n}}}{(3^n)^{\frac{1}{n}}} \right| = \lim_{n \to \infty} \sup \left| \frac{n^{\frac{1}{n}}}{3} \right| = \frac{1}{3} < 1$$

$\alpha = \frac{1}{3} < 1$ converge

Unit (4)

Sequence and series of $f^{\underline{n}}$

Power series

$$S = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$T = \sum a_n (x-a)^n = a_0 + a_1 (x-a) + a_2 (x-a)^2 + \cdots$$

S converges
- $x = 0$
- all x
- In bounded specific interval

Unit (4)

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Sequence and series of $f^{\underline{n}}$

Power series.

Theorem:

$$\sum a_n x^n \;,\; \beta = \limsup |a_n|^{1/n} \;,\; R = \frac{1}{\beta} \qquad \beta = \frac{1}{R}$$

Converges for $|x| < R$ — diverges for $|x| > R$

proof:

$$\alpha_x = \limsup \left| |a_n x^n|^{1/n} \right| = |x| \limsup |a_n|^{1/n}$$

$$\alpha_x = |x| \beta$$

$$\alpha_x = \frac{|x|}{R}$$

We consider three cases.

① At $0 < R < \infty$

$$\alpha_x = \frac{|x|}{R} = \begin{cases} < 1 \text{ when } |x| < R & \Rightarrow \text{ series is converges} \\ > 1 \text{ when } |x| > R & \Rightarrow \text{ " " diverges} \end{cases}$$

② At $R = 0$, $\alpha_x \to \infty > 1 \Rightarrow$ The series is diverges $\forall x \neq 0$

③ $R \to \infty \Rightarrow \alpha = 0 < 1 \Rightarrow$ " " " converge $\forall x$

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Exercise

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Consider the power series

$$\sum_{n=0}^{\infty} (x-2)^n$$

Find Radius and the interval of convergence

$\cancel{\sum a_n (x-a)^n} = (x-2)^n$

$$\frac{U_{n+1}}{U_n} = \frac{(x-2)^{n+1}}{(x-2)^n} = (x-2)$$

$$\alpha = \lim_{n \to \infty} \sup \frac{|x-2|}{1}$$

$\alpha < 1$

converges $\frac{|x-2|}{1} < 1 \Rightarrow |x-2| < 1 \Rightarrow -1 < x-2 < 1.$

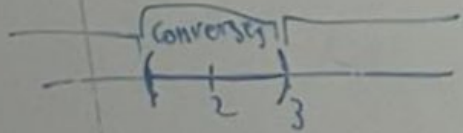
$1 < x < 3$

$(1, 3)$ interval of convergence

$$|x-2| > 1$$

$x-2 > 1 \Rightarrow x > 3$

$x-2 < -1 \Rightarrow x < 1$



In other way:

If we choose $\varepsilon = \frac{1}{3}$

$$|x^n - 0| < \frac{1}{3}$$

$$x^n < \frac{1}{3}$$

$$x < \frac{1}{\sqrt[n]{3}}$$

$x^n \to 0$ in $\left[0, \frac{1}{\sqrt[n]{3}}\right)$

$\therefore$ In the interval $\left[\frac{1}{\sqrt[n]{3}}, 1\right]$, $x^n \not\to 0$

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Uniform convergence

Remark

$\underbrace{f_1, f_2, \dots, f_n}_{f_4} \to f(x)$

$N = 4$

① In point wise convergence $N = N(\varepsilon, x)$

② In uniform convergence $N = N(\varepsilon)$

Example: $f_n(x) = x^n$ in $[0,1]$

$x_n \to 0$ $\quad [0,1)$

$$|f_n(x) - f(x)| < \varepsilon$$

$$|x^n - 0| < \varepsilon \Rightarrow x^n < \varepsilon$$

$$\ln x^n < \ln \varepsilon$$

$$n \ln x < \ln \varepsilon$$

$$n > \frac{\ln \varepsilon}{\ln x} \Rightarrow N = \frac{\ln \varepsilon}{\ln x} \Rightarrow N = N(\varepsilon, x)$$

$$f_n \xrightarrow{U} f$$

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Uniform convergence

Def^n:

Let $f_n(x)$ be asequence defined on aset $S \subseteq \mathbb{R}$ then we said that f_n converges uniformally to $f(x)$, if one of the following conditcón is satiesfied

(1) $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \ni$

$$|f_n(x) - f(x)| < \varepsilon, \ \forall n \geq N \text{ and } \forall x \in S$$

(2) $$\lim_{n \to \infty} \sup \{ |f_n(x) - f(x)| : x \in S \} = 0$$

when the derivative of $(f_n(x) - f(x))$ exist

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Uniform convergence

Example 2

let $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in R$

Prove that f_n does not converges uniformally in R

Soln : $\because \lim_{n\to\infty} f_n(x) = 0$

$f_n(x) - f(x) = \frac{nx}{1+n^2x^2} - 0 = \frac{nx}{1+n^2x^2}$

$\frac{d}{dx}\left(f_n(x) - f(x)\right) = \frac{d}{dx}\left(\frac{nx}{1+n^2x^2}\right)$

$= \frac{-nx(2n^2x) + (1+n^2x^2)(n)}{(1+n^2x^2)^2} = 0$

$= -2n^3x^2 + n + n^3x^2 = 0$

$-n^3x^2 + n = 0$

$n^3x^2 = n$

$x^2 = \frac{1}{n^2}$

$x = \pm\frac{1}{n}$

$f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{n \cdot \frac{1}{\cancel{n}}}{1 + \cancel{n^2} \cdot \frac{1}{\cancel{n^2}}}$

$= \frac{1}{2}$

$\lim \sup \{f_n(x) - f(x)\} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0$

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Uniform convergence

Example 2

Let $f_n(x) = \dfrac{x}{1+nx^2}$, $x \in R$

Prove that f_n converges uniformly in R

Soln : $\because \lim_{n\to\infty} f_n(x) = 0$

$$f_n(x) - f(x) = \frac{x}{1+nx^2} - 0 = \frac{x}{1+nx^2}$$

$$\frac{d}{dx}\left(f_n(x) - f(x)\right) = \frac{(1)(1+nx^2) - 2nx(x)}{(1+nx^2)^2} = 0$$

$$1+nx^2 - 2nx^2 = 0$$

$$1 - nx^2 = 0$$

$$nx^2 = 1$$

$$x = \pm \frac{1}{\sqrt{n}}$$

$$\therefore f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{\frac{1}{\sqrt{n}}}{1+n\left(\frac{1}{\sqrt{n}}\right)^2}$$

$$= \frac{\frac{1}{\sqrt{n}}}{2}$$

$$f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2\sqrt{n}}$$

$\lim \sup \{|f_n(x) - f(x)|\}$

$$\lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$$

$$\therefore f_n \xrightarrow{U} 0$$

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Uniform convergence

Example 21

let $f_n = \frac{1+2\cos^2 nx}{\sqrt{n}}$

Prove that f_n converges uniformally to $f(x)=0$ in $\mathbb{R}$

Soln

$|f_n(x)-f(x)|<\varepsilon$

$\left|\frac{1+2\cos^2 nx}{\sqrt{n}}-0\right|<\varepsilon$

$\left|\frac{1+2\cos^2 nx}{\sqrt{n}}\right|<\frac{1+2}{\sqrt{n}}<\boxed{\frac{3}{\sqrt{n}}<\varepsilon}$

$\frac{9}{n}<\varepsilon$

$n>\frac{9}{\varepsilon} \Rightarrow N=\frac{9}{\varepsilon} \Rightarrow N=N(\varepsilon) \Rightarrow f_n \xrightarrow{U} f$

In other way:

if we choose $\varepsilon=\frac{1}{3}$

$|x^n-0|<\frac{1}{3}$

$x^n<\frac{1}{3}$

$x<\frac{1}{\sqrt[n]{3}}$

$x^n \to 0$ in $[0, \frac{1}{\sqrt[n]{3}})$

$\therefore$ In the interval $[\frac{1}{\sqrt[n]{3}}, 1]$, $x^n \not\to 0$

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Example 8]

$f_n(x) = n^2 x^n (1-x)$, $x \in [0,1]$

(a) Find $f(x) = \lim f_n(x)$.

(b) Does f_n converges to $f(x)$ uniformally?

$\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$

$\left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n = e$

Soln

(a) clearly $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} n^2 x^n (1-x) = 0$, $\forall x \in [0,1]$

(b) $\frac{d}{dx}(f_n - f) = \frac{d}{dx}[n^2 x^n (1-x)] = n^2 \frac{d}{dx} x^n (1-x) -$

$= n^2 [x^n(-1) + (1-x) n x^{n-1}] = 0$

$-x^n + n x^{n-1} - n x^n = 0$

$-x^n(1+n) + n x^{n-1} = 0$

$(1+n) x^n = n x^{n-1}$

$x = \frac{n}{n+1}$

$\sup f_n\left(\frac{n}{n+1}\right) = n^2 \left(\frac{n}{n+1}\right)^n \overbrace{\left(1 - \frac{n}{n+1}\right)}^{\frac{n+1-n}{n+1}} = \frac{n^2}{n+1} \cdot \left(\frac{n}{n+1}\right)^n$

$= \frac{n^2}{n+1} \cdot \frac{1}{e}$

$\lim \sup |f_n - f| = \lim\limits_{n\to\infty} \frac{n^2}{n+1} \cdot \frac{1}{e} \longrightarrow \infty$

$\therefore f_n$ is not converges uniformally to 0 , $f_n \overset{U}{\not\to} f$