WHAT IS CONTINUUM MECHANICS?

Continuum mechanics studies the response of materials to different loading conditions. Its subject matter can be divided into two main parts:

(1) general principles common to all media and

(2) constitutive equations defining idealized materials.

WHAT IS CONTINUUM MECHANICS?

The general principles are axioms considered to be self-evident from our experience with the physical world, such as conservation of mass; the balance of linear momentum, moment of momentum, and energy; and the entropy inequality law.

WHAT IS CONTINUUM MECHANICS?

Mathematically, there are two equivalent forms of the general principles: (1) the integral form, formulated for a finite volume of material in the continuum, and (2) the field equations for differential volume of material (particles) at every point of the field of interest.

TENSORS

PART A: INDICIAL NOTATION

Summation Convention, Dummy Indices

Consider the sum

$$s = a_1x_1 + a_2x_2 + \ldots + a_nx_n.$$

We can write the preceding equation in a compact form using a summation sign:

$$s = \sum_{i=1}^{n} a_ix_i.$$

The following equations have exactly the same meaning

$$s = \sum_{j=1}^{n} a_jx_j, \quad s = \sum_{m=1}^{n} a_mx_m, \quad s = \sum_{k=1}^{n} a_kx_k.$$

> ***Whenever an index is repeated once, it is a dummy index indicating a summation with the index running through the integral numbers 1, 2, . . ., n.***

Einstein's Summation Convention

$$s = a_ix_i \quad \text{or} \quad s = a_jx_j \quad \text{or} \quad s = a_mx_m$$

$$s = \sum_{i=1}^{n} a_ix_i. \quad s = \sum_{j=1}^{n} a_jx_j, \quad s = \sum_{m=1}^{n} a_mx_m$$

Note that: an index should never be repeated more than once when the summation convention is used.

$$a_ib_ix_i \text{ or } a_mb_mx_m$$

are not defined within this convention.

$$\sum_{i=1}^{n} a_ib_ix_i,$$

must retain its summation sign.

Example: If n=3, then

$$a_i x_i = a_1 x_1 + a_2 x_2 + a_3 x_3,$$

The summation convention obviously can be used to express a double sum, a triple sum, and so on. For example, we can write:

$$\alpha = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i x_j$$

concisely as

$$\alpha = a_{ij} x_i x_j.$$

$$\alpha = a_{ij} x_i x_j = a_{11} x_1 x_1 + a_{12} x_1 x_2 + a_{13} x_1 x_3 + a_{21} x_2 x_1 + a_{22} x_2 x_2 + a_{23} x_2 x_3 + a_{31} x_3 x_1 + a_{32} x_3 x_2 + a_{33} x_3 x_3.$$

first, sum over i, and then sum over j (or vice versa), i.e.,

$$a_{ij} x_i x_j = a_{1j} x_1 x_j + a_{2j} x_2 x_j + a_{3j} x_3 x_j,$$

where

$$a_{1j} x_1 x_j = a_{11} x_1 x_1 + a_{12} x_1 x_2 + a_{13} x_1 x_3,$$

Similarly, the indicial notation $a_{ijk} x_i x_j x_k$ represents a triple sum of 27 terms, that is,

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_{ijk} x_i x_j x_k = a_{ijk} x_i x_j x_k.$$

Free Indices

A free index is one that appears in every expression of an equation, except for expressions that contain real numbers (scalars) only.

Consider the following system of three equations:

$$x'_1 = a_{11} x_1 + a_{12} x_2 + a_{13} x_3,$$
$$x'_2 = a_{21} x_1 + a_{22} x_2 + a_{23} x_3,$$
$$x'_3 = a_{31} x_1 + a_{32} x_2 + a_{33} x_3.$$

using the summation convention, it can be written as:

$$x'_1 = a_{1m} x_m,$$
$$x'_2 = a_{2m} x_m,$$
$$x'_3 = a_{3m} x_m,$$

which can be shortened into → $x'_i = a_{im}\, x_m, \quad i = 1, 2, 3.$

Example: If n=3, $F_i = A_i B_j C_j$ then

$F_1 = A_1 (B_1C_1 + B_2C_2 + B_3C_3)$

$F_2 = A_2 (B_1C_1 + B_2C_2 + B_3C_3)$

$F_3 = A_3 (B_1C_1 + B_2C_2 + B_3C_3)$

HW1: $G_k = H_k (2 - 3\, A_i B_i) + P_j Q_j F_k$

HW2: $A_i = 2 + B_i + C_i + D_i + (F_j G_j - H_j P_j)\, E_i$

If there are two free indices appearing in an equation such as:

$$T_{ij} = A_{im} A_{jm},$$

then the equation is a shorthand for the nine equations, each with a sum of three terms on the right-hand side.

HW: Expand $T_{ij} = A_{im} A_{jm},$

The Kronecker Delta

The Kronecker delta, denoted by δ_{ij}, is defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

That is,

$$\delta_{11} = \delta_{22} = \delta_{33} = 1, \quad \delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0.$$

$$[\delta_{ij}] = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notes on Kronecker Delta

1. $\delta_{ij} = \delta_{ji}$

2. $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33}$

$= 1+1+1$

$=3$

3.

$$\delta_{1m}a_m = \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 = \delta_{11}a_1 = a_1,$$
$$\delta_{2m}a_m = \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 = \delta_{22}a_2 = a_2,$$
$$\delta_{3m}a_m = \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 = \delta_{33}a_3 = a_3,$$

$$\delta_{im}a_m = a_i.$$

4.

$$\delta_{1m}T_{mj} = \delta_{11}T_{1j} + \delta_{12}T_{2j} + \delta_{13}T_{3j} = T_{1j},$$
$$\delta_{2m}T_{mj} = \delta_{21}T_{1j} + \delta_{22}T_{2j} + \delta_{23}T_{3j} = T_{2j},$$
$$\delta_{3m}T_{mj} = \delta_{31}T_{1j} + \delta_{32}T_{2j} + \delta_{33}T_{3j} = T_{3j},$$

$$\delta_{im}T_{mj} = T_{ij}.$$

In particular

$$\delta_{im}\delta_{mj} = \delta_{ij}, \quad \delta_{im}\delta_{mn}\delta_{nj} = \delta_{ij}, \quad \text{etc.}$$

Example: Simplify $\delta_{ij}\delta_{kj}\delta_{in}$

$$\delta_{ij}\delta_{kj}\delta_{in} = \delta_{ij}\delta_{jk}\delta_{in}$$
$$= \delta_{ik}\delta_{in}$$
$$= \delta_{ki}\delta_{in}$$
$$= \delta_{kn}$$

5. If $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors perpendicular to one another, then clearly,

The Permutation Symbol

The Permutation symbol, denoted by ε_{ijk}, is defined by:

$$\varepsilon_{ijk} = \left\{ \begin{matrix} 1 \\ -1 \\ 0 \end{matrix} \right\} \equiv \text{according to whether } i,j,k \begin{pmatrix} \text{form an even} \\ \text{form an odd} \\ \text{do not form} \end{pmatrix} \text{permutation of } 1,2,3,$$

We note that

$$\begin{aligned} \varepsilon_{123} &= \varepsilon_{231} = \varepsilon_{312} = +1, \\ \varepsilon_{213} &= \varepsilon_{321} = \varepsilon_{132} = -1, \\ \varepsilon_{111} &= \varepsilon_{112} = \varepsilon_{222} = \ldots = 0. \end{aligned}$$

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{jik} = -\varepsilon_{kji} - \varepsilon_{ikj}.$$

If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a right-handed triad, then

$$\begin{aligned} &\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3\,, \\ &\mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3\,, \\ &\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1\,, \\ &\mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1\,, \\ &\ldots \end{aligned}$$

Which can be written in a short form as

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\, \mathbf{e}_\mathbf{k} = \varepsilon_{jki}\, \mathbf{e}_\mathbf{k} = \varepsilon_{kij}\, \mathbf{e}_\mathbf{k}$$

Now, if $\mathbf{a}=a_i\mathbf{e}_i$ and $\mathbf{b}=b_i\mathbf{e}_i$, then, since the cross-product is distributive, we have

$$\mathbf{a} \times \mathbf{b} = (a_i\mathbf{e}_i) \times (b_j\mathbf{e}_j) = a_i b_j(\mathbf{e}_i \times \mathbf{e}_j) = a_i b_j \varepsilon_{ijk}\mathbf{e}_k$$

HW: Prove that

$$\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

INDICIAL NOTATION MANIPULATIONS

Substitution: if

$$a_i = U_{im} b_m, \quad \text{(i)}$$

and

$$b_i = V_{im} c_m \quad \text{(ii)}$$

then, in order to substitute the b_i in Eq. (ii) into the b_m in Eq. (i), we must first change the free index in Eq. (ii) from i to m and the dummy index m to some other letter—say, n—so that $b_m = V_{mn} c_n$

$$a_i = U_{im} V_{mn} c_n \quad \text{(iii)}$$

Note that Eq. (iii) represents three equations, each having a sum of nine terms on its right-hand side.

INDICIAL NOTATION MANIPULATIONS

Multiplication: if

$$p = a_m b_m \qquad \text{and} \qquad q = c_m d_m$$

then,

$$pq = a_m b_m c_n d_n$$

It is important to note that $pq \neq a_m b_m c_m d_m$. In fact, the right-hand side of this expression, i.e., $a_m b_m c_m d_m$, is not even defined in the summation convention, and further, it is obvious that

$$pq \neq \sum_{m=1}^{3} a_m b_m c_m d_m$$

Since the dot product of vectors is distributive, therefore, if a $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_i\mathbf{e}_i$, then

$$\mathbf{a} \cdot \mathbf{b} = (a_i\mathbf{e}_i) \cdot (b_j\mathbf{e}_j) = a_i b_j(\mathbf{e}_i \cdot \mathbf{e}_j)$$

In particular, if $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors perpendicular to one another, then $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ so that

$$\mathbf{a} \cdot \mathbf{b} = a_i b_j \delta_{ij} = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

which is the familiar expression for the evaluation of the dot product in terms of the vector components.

INDICIAL NOTATION MANIPULATIONS

Factoring: if

$$T_{ij}n_j - \lambda n_i = 0,$$

then, Using Kronecker delta, we can write $n_i = \delta_{ij}n_j$, so that we have

$$T_{ij}n_j - \lambda\delta_{ij}n_j = 0.$$

Thus,

$$(T_{ij} - \lambda\delta_{ij})n_j = 0.$$

INDICIAL NOTATION MANIPULATIONS

Contraction: The operation of identifying two indices is known as a *contraction*. Contraction indicates a sum on the index. For example, T_{ii} is the contraction of T_{ij} with

$$T_{ii} = T_{11} + T_{22} + T_{33}.$$

If

$$T_{ij} = \lambda\Delta\delta_{ij} + 2\mu E_{ij}$$

then

$$T_{ii} = \lambda\Delta\delta_{ii} + 2\mu E_{ii} = 3\lambda\Delta + 2\mu E_{ii}$$

Problems for Part A

1. Given

$$[S_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix} \text{ and } [a_i] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

evaluate (a) S_{ii}, (b) $S_{ij}S_{ij}$, (c) $S_{ji}S_{ji}$, (d) $S_{jk}S_{kj}$, (e) $a_m a_m$, (f) $S_{mn}a_m a_n$, and (g) $S_{nm}a_m a_n$.

2. Determine which of these equations has an identical meaning with $a_i = Q_{ij}a_j'$

(a) $a_p=Q_{pm}a_m'$
(b) $a_p=Q_{qp}a_q'$
(c) $a_m=a_n'Q_{mn}$

3. Given the following matrices

$$[a_i] = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, [B_{ij}] = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

demonstrate the equivalence of subscripted equations and corresponding matrix equations in the following two problems

(a) $b_i=B_{ij}a_j$ and $[b]=[B][a]$
(b) $s=B_{ij}a_i a_j$ and $s=[a]^T[B][a]$.

TENSORS

PART B: TENSORS: A LINEAR TRANSFORMATION

Tensors: A Linear Transformation

Let **T** be a transformation that transforms any vector into another vector. If **T** transforms **a** into **c** and **b** into **d**, we write **Ta** = **c** and **Tb** = **d**.

If **T** has the following linear properties:

$$\mathbf{T}(\mathbf{a} + \mathbf{b}) = \mathbf{Ta} + \mathbf{Tb}$$

$$\mathbf{T}(\alpha\mathbf{a}) = \alpha\mathbf{Ta}$$

where **a** and **b** are two arbitrary vectors and α is an arbitrary scalar, then **T** is called a *linear transformation*. It is also called a *second-order tensor* or simply a *tensor*. 1

An alternative and equivalent definition of a linear transformation is given by the single linear property:

$$\mathbf{T}(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\mathbf{Ta} + \beta\mathbf{Tb}$$

where **a** and **b** are two arbitrary vectors and α and β are arbitrary scalars.

If two tensors, **T** and **S**, transform any arbitrary vector a identically, these two tensors are the same, that is, if **Ta = Sa** for any **a**, then **T = S**.

We note, however, that two different tensors may transform specific vectors identically.

Example 1

Let **T** be a nonzero transformation that transforms every vector into a fixed nonzero vector **n**. Is this transformation a tensor?

Let **a** and **b** be any two vectors; then **Ta = n** and **Tb = n**.

Since **a + b** is also a vector, therefore **T(a + b) = n**.

Clearly **T(a + b)** does not equal **Ta** + **Tb**.

Thus, this transformation is not a linear one. In other words, it is not a tensor.

Example 2

Let **T** be a transformation that transforms every vector into a vector that is k times the original vector. Is this transformation a tensor?

Let **a** and **b** be arbitrary vectors and α and β be arbitrary scalars; then, by the definition of **T**,

$$\mathbf{Ta} = k\mathbf{a},\ \mathbf{Tb} = k\mathbf{b} \quad \text{and } \mathbf{T}(\alpha\mathbf{a} + \beta\mathbf{b}) = k(\alpha\mathbf{a} + \beta\mathbf{b})$$

Clearly,

$$\mathbf{T}(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha k\mathbf{a} + \beta k\mathbf{b} = \alpha\mathbf{Ta} + \beta\mathbf{Tb}$$

Therefore, **T** is a linear transformation. In other words, it is a tensor.

If $k = 0$, then the tensor transforms all vectors into a zero vector (null vector). This tensor is the *zero tensor* or *null tensor* and is symbolized by the boldface **0**.

Example 3

Let **T** be a tensor that transforms the specific vectors **a** and **b** as follows:

$$\mathbf{Ta} = \mathbf{a} + 2\mathbf{b};$$

$$\mathbf{Tb} = \mathbf{a} - \mathbf{b}.$$

Given a vector $\mathbf{c} = 2\mathbf{a} + \mathbf{b}$, find **Tc.**

Using the linearity property of tensors, we have

$$\mathbf{Tc} = \mathbf{T}(2\mathbf{a} + \mathbf{b}) = 2\mathbf{Ta} + \mathbf{Tb} = 2(\mathbf{a} + 2\mathbf{b}) + (\mathbf{a} - \mathbf{b}) = 3\mathbf{a} + 3\mathbf{b}$$

COMPONENTS OF A TENSOR

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be unit vectors in the direction of the x_1-, x_2-, x_3-, respectively, of a rectangular Cartesian coordinate system. Under a transformation $\mathbf{T}$, these vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ become $\mathbf{Te}_1$; $\mathbf{Te}_2$; $\mathbf{Te}_3$. Each of these $\mathbf{Te}_i$, being a vector, can be written as:

$$\mathbf{Te}_1 = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3,$$
$$\mathbf{Te}_2 = T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3,$$
$$\mathbf{Te}_3 = T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2 + T_{33}\mathbf{e}_3,$$

$$\mathbf{Te}_i = T_{ji}\mathbf{e}_j.$$

$\mathbf{Te}_i = T_{ji}\mathbf{e}_j$

COMPONENTS OF A TENSOR

The components T_{ij} in the preceding equations are defined as the components of the tensor $\mathbf{T}$. These components can be put in a matrix as follows:

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}.$$

$$\mathbf{Te}_1 = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3$$
$$\mathbf{Te}_2 = T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3$$
$$\mathbf{Te}_3 = T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2 + T_{33}\mathbf{e}_3$$

This matrix is called the *matrix of the tensor* $\mathbf{T}$ with respect to the set of base vectors $\{\mathbf{e}_i\}$.

- The elements of the first column in the matrix are components of the vector $\mathbf{Te}_1$
- Those in the second column are the components of the vector $\mathbf{Te}_2$,
- And those in the third column are the components of $\mathbf{Te}_3$.

$\mathbf{Te}_i = T_{ji}\mathbf{e}_j$

Example 4

Obtain the matrix for the tensor $\mathbf{T}$ that transforms the base vectors as follows:

$$\mathbf{Te}_1 = 4\mathbf{e}_1 + \mathbf{e}_2,$$
$$\mathbf{Te}_2 = 2\mathbf{e}_1 + 3\mathbf{e}_3,$$
$$\mathbf{Te}_3 = -\mathbf{e}_1 + 3\mathbf{e}_2 + \mathbf{e}_3.$$

$$[\mathbf{T}] = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}.$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

Example 5

Let **T** transform every vector into its mirror image with respect to a fixed plane; if $\mathbf{e}_1$ is normal to the reflection plane ($\mathbf{e}_2$ and $\mathbf{e}_3$ are parallel to this plane), find a matrix of **T**.

Since the normal to the reflection plane is transformed into its negative and vectors parallel to the plane are not altered, we have

$$\mathbf{T}\mathbf{e}_1 = -\mathbf{e}_1, \quad \mathbf{T}\mathbf{e}_2 = \mathbf{e}_2, \quad \mathbf{T}\mathbf{e}_3 = \mathbf{e}_3$$

$$\mathbf{T}\mathbf{e}_1 = -\mathbf{e}_1, \quad \mathbf{T}\mathbf{e}_2 = \mathbf{e}_2, \quad \mathbf{T}\mathbf{e}_3 = \mathbf{e}_3$$

which corresponds to

$$[\mathbf{T}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i}$$

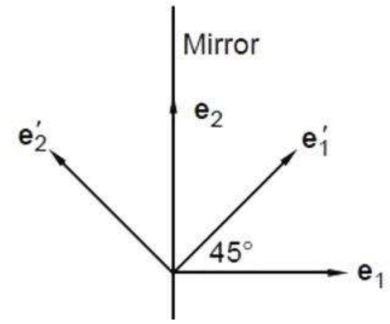
We note that this is only one of the infinitely many matrices of the tensor **T**; each depends on a particular choice of base vectors.

In the preceding matrix, the choice of $\mathbf{e}_i$ is indicated at the bottom-right corner of the matrix.

If we choose $\mathbf{e}'_1$ and $\mathbf{e}'_2$ to be on a plane perpendicular to the mirror, with each making 45° with the mirror, as shown in figure below, and $\mathbf{e}'_3$ pointing straight out from the paper, then we have

$$\mathbf{T}\mathbf{e}'_1 = \mathbf{e}'_2, \quad \mathbf{T}\mathbf{e}'_2 = \mathbf{e}'_1, \quad \mathbf{T}\mathbf{e}'_3 = \mathbf{e}'_3$$

$$[\mathbf{T}]' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}'_i}.$$



$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

Note

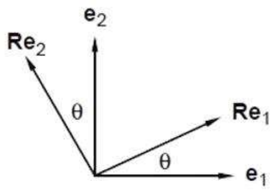
Throughout this course, we denote the matrix of a tensor **T** with respect to the basis $\{\mathbf{e}_i\}$ by either [**T**] or $[T_{ij}]$ and with respect to the basis $\{\mathbf{e}'_i\}$ by either $[\mathbf{T}]'$ or $[T'_{ij}]$.

The last two matrices should not be confused with $[\mathbf{T}']$, which represents the matrix of the tensor $\mathbf{T}'$ with respect to the basis $\{\mathbf{e}_i\}$, not the matrix of **T** with respect to the primed basis $\{\mathbf{e}'_i\}$.

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

Example 6

From the figure below, it is clear that



$$\mathbf{Re}_1 = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2,$$
$$\mathbf{Re}_2 = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2,$$
$$\mathbf{Re}_3 = \mathbf{e}_3.$$

which corresponds to

$$[\mathbf{R}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i}.$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

Since $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0$ (because they are mutually perpendicular), it can be easily verified from Eq.

$$\mathbf{T}\mathbf{e}_1 = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3,$$
$$\mathbf{T}\mathbf{e}_2 = T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3,$$
$$\mathbf{T}\mathbf{e}_3 = T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2 + T_{33}\mathbf{e}_3,$$

that

$$T_{11} = \mathbf{e}_1 \cdot \mathbf{T}\mathbf{e}_1, \quad T_{12} = \mathbf{e}_1 \cdot \mathbf{T}\mathbf{e}_2, \quad T_{13} = \mathbf{e}_1 \cdot \mathbf{T}\mathbf{e}_3,$$
$$T_{21} = \mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_1, \quad T_{22} = \mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_2, \quad T_{23} = \mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_3,$$
$$T_{31} = \mathbf{e}_3 \cdot \mathbf{T}\mathbf{e}_1, \quad T_{32} = \mathbf{e}_3 \cdot \mathbf{T}\mathbf{e}_2, \quad T_{33} = \mathbf{e}_3 \cdot \mathbf{T}\mathbf{e}_3,$$

or

$$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

These equations are totally equivalent to:

$$\begin{aligned}\mathbf{T}\mathbf{e}_1 &= T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3,\\ \mathbf{T}\mathbf{e}_2 &= T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3,\\ \mathbf{T}\mathbf{e}_3 &= T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2 + T_{33}\mathbf{e}_3,\end{aligned} \quad \text{or} \quad \mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j.$$

and can also be regarded as the definition of the components of a tensor **T**.

They are often more convenient to use than $\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$.

We note again that the components of a tensor depend on the coordinate systems through the set of base vectors. Thus,

$$T'_{ij} = \mathbf{e}'_i \cdot \mathbf{T}\mathbf{e}'_j,$$

where T'_{ij} are the components of the same tensor **T** with respect to the base vectors $\{\mathbf{e}'_i\}$. It is important to note that vectors and tensors are independent of coordinate systems, but their *components* are dependent on the coordinate systems.

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

COMPONENTS OF A TRANSFORMED VECTOR

Given the vector **a** and the tensor **T**, which transforms **a** into **b** (i.e., **b = Ta**), we wish to compute the components of **b** from the components of **a** and the components of **T**.

Let the components of **a** with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be (a_1, a_2, a_3), that is,

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$$

then

$$\mathbf{b} = \mathbf{T}\mathbf{a} = \mathbf{T}(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = a_1\mathbf{T}\mathbf{e}_1 + a_2\mathbf{T}\mathbf{e}_2 + a_3\mathbf{T}\mathbf{e}_3$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

$$\begin{aligned}b_1 &= \mathbf{b}\cdot\mathbf{e}_1 = \mathbf{e}_1\cdot\mathbf{T}(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = a_1(\mathbf{e}_1\cdot\mathbf{T}\mathbf{e}_1) + a_2(\mathbf{e}_1\cdot\mathbf{T}\mathbf{e}_2) + a_3(\mathbf{e}_1\cdot\mathbf{T}\mathbf{e}_3)\\ b_2 &= \mathbf{b}\cdot\mathbf{e}_2 = \mathbf{e}_2\cdot\mathbf{T}(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = a_1(\mathbf{e}_2\cdot\mathbf{T}\mathbf{e}_1) + a_2(\mathbf{e}_2\cdot\mathbf{T}\mathbf{e}_2) + a_3(\mathbf{e}_2\cdot\mathbf{T}\mathbf{e}_3)\\ b_3 &= \mathbf{b}\cdot\mathbf{e}_3 = \mathbf{e}_3\cdot\mathbf{T}(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = a_1(\mathbf{e}_3\cdot\mathbf{T}\mathbf{e}_1) + a_2(\mathbf{e}_3\cdot\mathbf{T}\mathbf{e}_2) + a_3(\mathbf{e}_3\cdot\mathbf{T}\mathbf{e}_3)\end{aligned}$$

$$\begin{aligned}b_1 &= T_{11}a_1 + T_{12}a_2 + T_{13}a_3\\ b_2 &= T_{21}a_1 + T_{22}a_2 + T_{23}a_3\\ b_3 &= T_{31}a_1 + T_{32}a_2 + T_{33}a_3\end{aligned} \qquad \text{.......} \ (1)$$

We can write the preceding three equations in matrix form as:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

$$[\mathbf{b}] = [\mathbf{T}][\mathbf{a}].$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

We can also derive Eq. (1) using indicial notations as follows:
From $\mathbf{a} = a_i\mathbf{e}_i$, we get $\mathbf{Ta} = \mathbf{T}(a_i\mathbf{e}_i) = a_i\mathbf{T}\mathbf{e}_i$.

Since $\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$, $\mathbf{b} = \mathbf{Ta} = a_iT_{ji}\mathbf{e}_j$ so that

$$b_m = \mathbf{b}\cdot\mathbf{e}_m = a_iT_{ji}\mathbf{e}_j\cdot\mathbf{e}_m = a_iT_{ji}\delta_{jm} = a_iT_{mi};$$

that is,

$$b_m = a_iT_{mi} = T_{mi}a_i \,. \qquad \text{……. (2)}$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

Example

Given that a tensor **T** transforms the base vectors as follows:

$$\mathbf{T}\mathbf{e}_1 = 2\mathbf{e}_1 - 6\mathbf{e}_2 + 4\mathbf{e}_3,$$
$$\mathbf{T}\mathbf{e}_2 = 3\mathbf{e}_1 + 4\mathbf{e}_2 - 1\mathbf{e}_3,$$
$$\mathbf{T}\mathbf{e}_3 = -2\mathbf{e}_1 + 1\mathbf{e}_2 + 2\mathbf{e}_3.$$

how does this tensor transform the vector $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$?

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

Solution:

Use the matrix equation

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -2 \\ -6 & 4 & 1 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

we obtain $\mathbf{b} = 2\mathbf{e}_1 + 5\mathbf{e}_2 + 8\mathbf{e}_3$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

SUM OF TENSORS

Let **T** and **S** be two tensors. The sum of **T** and **S**, denoted by **T**+**S**, is defined by

$$(\mathbf{T}+\mathbf{S})\mathbf{a} = \mathbf{Ta} + \mathbf{Sa}$$

for any vector **a**. It is easily seen that **T**+**S**, so defined, is indeed a tensor.

To find the components of **T**+**S**, let

$$\mathbf{W} = \mathbf{T}+\mathbf{S}.$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

The components of **W** are

$$W_{ij} = \mathbf{e}_i \cdot (\mathbf{T}+\mathbf{S})\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{S}\mathbf{e}_j,$$

that is,

$$W_{ij} = T_{ij} + S_{ij}$$

In matrix notation, we have

$$[\mathbf{W}] = [\mathbf{T}] + [\mathbf{S}],$$

and that the tensor sum is consistent with the matrix sum.

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

PRODUCT OF TENSORS

Let **T** and **S** be two tensors and **a** be an arbitrary vector. Then **TS** and **ST** are defined to be the transformations (**HW**: Prove that **TS** and **ST** are both tensors) such that

$$(\mathbf{TS})\mathbf{a} = \mathbf{T}(\mathbf{Sa}) \quad \text{and} \quad (\mathbf{ST})\mathbf{a} = \mathbf{S}(\mathbf{Ta}).$$

The components of **TS** are

$$(\mathbf{TS})_{ij} = \mathbf{e}_i \cdot (\mathbf{TS})\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{T}(\mathbf{Se}_j) = \mathbf{e}_i \cdot \mathbf{T} S_{mj}\mathbf{e}_m = S_{mj}\mathbf{e}_i \cdot \mathbf{Te}_m = S_{mj}T_{im}$$

that is, $(\mathbf{TS})_{ij} = T_{im}S_{mj}$

Similarly, $(\mathbf{ST})_{ij} = S_{im}T_{mj}$

These are equivalent to matrix equation:

$$[\mathbf{TS}] = [\mathbf{T}][\mathbf{S}] \quad \text{and} \quad [\mathbf{ST}] = [\mathbf{S}][\mathbf{T}].$$

The two products are, in general, different. Thus, it is clear that in general $\mathbf{TS} \neq \mathbf{ST}$. That is, in general, the tensor product is not commutative.

If **T**, **S**, and **V** are three tensors, then, by repeatedly using the definition, we have

$$(\mathbf{T}(\mathbf{SV}))\mathbf{a} \equiv \mathbf{T}((\mathbf{SV})\mathbf{a}) \equiv \mathbf{T}(\mathbf{S}(\mathbf{Va})) \quad \text{and} \quad (\mathbf{TS})(\mathbf{Va}) \equiv \mathbf{T}(\mathbf{S}(\mathbf{Va})),$$

that is

$$\mathbf{T}(\mathbf{SV}) = \mathbf{TS}(\mathbf{V}) = \mathbf{TSV}.$$

Thus, the tensor product is associative. It is, therefore, natural to define the integral positive powers of a tensor by these simple products, so that

$$\mathbf{T}^2 = \mathbf{TT}, \quad \mathbf{T}^3 = \mathbf{TTT}, \ldots.$$

Example

(a) Let **R** correspond to a 90° right-hand rigid body rotation about the x_3-axis. Find the matrix of **R**.

(b) Let **S** correspond to a 90° right-hand rigid body rotation about the x_1-axis. Find the matrix of **S**.

(c) Find the matrix of the tensor that corresponds to the rotation **R**, followed by **S**.

(d) Find the matrix of the tensor that corresponds to the rotation **S**, followed by **R**.

(e) Consider a point P whose initial coordinates are (1,1,0). Find the new position of this point after the rotations of part (c). Also find the new position of this point after the rotations of part (d).

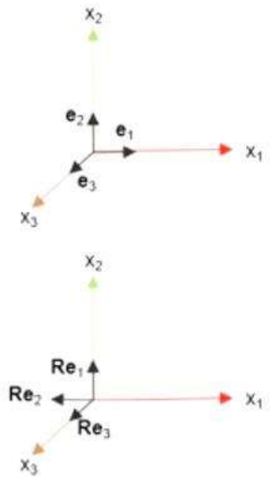
Example

(a) Let **R** correspond to a 90° right-hand rigid body rotation about the x_3-axis. Find the matrix of **R**.

$$\mathbf{R}\mathbf{e}_1 = \mathbf{e}_2,\ \mathbf{R}\mathbf{e}_2 = -\mathbf{e}_1,\ \mathbf{R}\mathbf{e}_3 = \mathbf{e}_3$$

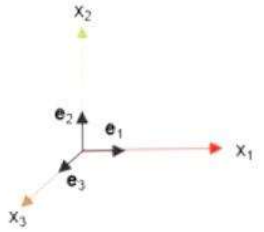
that is,

$$[\mathbf{R}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Example

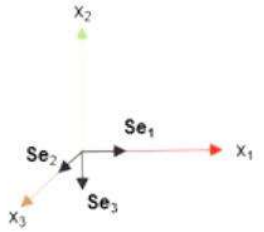
(b) Let **S** correspond to a 90° right-hand rigid body rotation about the x_1-axis. Find the matrix of **S**.



$$\mathbf{S}\mathbf{e}_1 = \mathbf{e}_1,\ \mathbf{S}\mathbf{e}_2 = \mathbf{e}_3,\ \mathbf{S}\mathbf{e}_3 = -\mathbf{e}_2$$

that is,

$$[\mathbf{S}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$



Example

(c) Find the matrix of the tensor that corresponds to the rotation **R**, followed by S.

$$\mathbf{S}(\mathbf{R}\mathbf{a}) = (\mathbf{S}\mathbf{R})\mathbf{a}$$

that is,

$$[\mathbf{SR}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

Example

(d) Find the matrix of the tensor that corresponds to the rotation **S**, followed by **R**.

$$\mathbf{R}(\mathbf{S}\mathbf{a}) = (\mathbf{R}\mathbf{S})\mathbf{a}$$

that is,

$$[\mathbf{RS}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Example

(e) Consider a point P whose initial coordinates are (1,1,0). Find the new position of this point after the rotations of part (c). Also find the new position of this point after the rotations of part (d).

Let **r** be the initial position of the material point P. Let **r*** and **r**** be the rotated position of P after the rotations of part (c) and part (d), respectively. Then

$$[\mathbf{r}^*] = [\mathbf{SR}][\mathbf{r}] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

that is,

$$\mathbf{r}^* = -\mathbf{e}_1 + \mathbf{e}_3$$

Example

(e) Consider a point P whose initial coordinates are (1,1,0). Find the new position of this point after the rotations of part (c). Also find the new position of this point after the rotations of part (d).

Let **r** be the initial position of the material point P. Let **r*** and **r**** be the rotated position of P after the rotations of part (c) and part (d), respectively. Then

$$[\mathbf{r}^{**}] = [\mathbf{RS}][\mathbf{r}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

that is,

$$\mathbf{r}^{**} = \mathbf{e}_2 + \mathbf{e}_3$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

TRANSPOSE OF TENSORS

The transpose of a tensor **T**, denoted by $\mathbf{T}^T$, is defined to be the tensor that satisfies the following identity for all vectors **a** and **b**:

$$\mathbf{a}\cdot\mathbf{T}\mathbf{b} = \mathbf{b}\cdot\mathbf{T}^T\mathbf{a}$$

It can be easily seen that $\mathbf{T}^T$ is a tensor (**HW**). From the preceding definition, we have

$$\mathbf{e}_j\cdot\mathbf{T}\mathbf{e}_i = \mathbf{e}_i\cdot\mathbf{T}^T\mathbf{e}_j$$

Thus,

$$\mathbf{T}_{ji} = \mathbf{T}^T{}_{ij}$$

or,

$$[\mathbf{T}]^T = [\mathbf{T}^T]$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

that is, the matrix of $\mathbf{T}^T$ is the transpose of the matrix T. We also note that

$$\mathbf{a}\cdot\mathbf{T}^T\mathbf{b} = \mathbf{b}\cdot(\mathbf{T}^T)^T\mathbf{a}$$

Thus, $\mathbf{b}\cdot\mathbf{T}\mathbf{a} = \mathbf{b}\cdot(\mathbf{T}^T)^T\mathbf{a}$ for any **a** and **b**, so that

$$(\mathbf{T}^T)^T = \mathbf{T}$$

It can be easily established that (**HW**)

$$(\mathbf{TS})^T = \mathbf{S}^T\mathbf{T}^T$$

That is, the transpose of a product of the tensors is equal to the product of transposed tensors in reverse order, which is consistent with the equivalent matrix identity. More generally,

$$(\mathbf{ABC}\ldots\mathbf{D})^T = \mathbf{D}^T\ldots\mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$$

DYADIC PRODUCT OF VECTORS

The dyadic product of vectors **a** and **b**, denoted by **ab**, is defined to be the transformation that transforms any vector **c** according to the rule:

$$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}).$$

Now, for any vectors **c**, **d**, and any scalars α and β, we have, from the preceding rule,

$$(\mathbf{ab})(\alpha\mathbf{c}+\beta\mathbf{d})=\mathbf{a}(\mathbf{b}\cdot(\alpha\mathbf{c}+\beta\mathbf{d}))=\mathbf{a}((\alpha\mathbf{b}\cdot\mathbf{c})+(\beta\mathbf{b}\cdot\mathbf{d}))=\alpha\mathbf{a}(\mathbf{b}\cdot\mathbf{c})+\beta\mathbf{a}(\mathbf{b}\cdot\mathbf{d})$$
$$=\alpha(\mathbf{ab})\mathbf{c}+\beta(\mathbf{ab})\mathbf{d}.$$

Thus, the dyadic product **ab** is a linear transformation.

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

Let **W** = **ab**, then the components of **W** are:

$$W_{ij} = \mathbf{e}_i \cdot \mathbf{W}\mathbf{e}_j = \mathbf{e}_i \cdot (\mathbf{ab})\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{a}(\mathbf{b} \cdot \mathbf{e}_j) = a_i b_j$$

$$W_{ij} = a_i b_j$$

or,

$$[\mathbf{W}] = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1 \quad b_2 \quad b_3]$$

In particular, the dyadic products of the base vectors $\mathbf{e}_i$ are:

$$[\mathbf{e}_1\mathbf{e}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [\mathbf{e}_1\mathbf{e}_2] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

$\mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j.$

Now, for **T**:

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & T_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \ldots$$

Thus, it is clear that any tensor T can be expressed as:

$$\mathbf{T} = T_{11}\mathbf{e}_1\mathbf{e}_1 + T_{12}\mathbf{e}_1\mathbf{e}_2 + T_{13}\mathbf{e}_1\mathbf{e}_3 + T_{21}\mathbf{e}_2\mathbf{e}_1 + \ldots = T_{ij}\mathbf{e}_i\mathbf{e}_j$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

$\mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j.$

TRACE OF A TENSOR

The trace of a tensor is a scalar that obeys the following rules: For any tensor **T** and **S** and any vectors **a** and **b**,

$$\text{tr}(\mathbf{T} + \mathbf{S}) = \text{tr}\,\mathbf{T} + \text{tr}\,\mathbf{S},$$

$$\text{tr}(\alpha\mathbf{T}) = \alpha\text{tr}\,\mathbf{T},$$

$$\text{tr}(\mathbf{ab}) = \mathbf{a} \cdot \mathbf{b}.$$

In terms of tensor components

$$\text{tr}\,\mathbf{T} = \text{tr}(T_{ij}\mathbf{e}_i\mathbf{e}_j) = T_{ij}\text{tr}(\mathbf{e}_i\mathbf{e}_j) = T_{ij}\mathbf{e}_i \cdot \mathbf{e}_j = T_{ij}\delta_{ij} = T_{ii}$$

That is, $\text{tr}\,\mathbf{T} = T_{11} + T_{22} + T_{33}$ = sum of diagonal elements.

It is, therefore, obvious that $\text{tr}\,\mathbf{T}^T = \text{tr}\,\mathbf{T}$.

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

$\mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j.$

Example

Show that for any second-order tensor **A** and **B**

$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$$

Let $\mathbf{C} = \mathbf{AB}$, then $C_{ij} = A_{im}B_{mj}$, so that $\text{tr}(\mathbf{AB}) = \text{tr}\,\mathbf{C} = C_{ii} = A_{im}B_{mi}$.

Let $\mathbf{D} = \mathbf{BA}$, then $D_{ij} = B_{im}A_{mj}$, so that $\text{tr}(\mathbf{BA}) = \text{tr}\,\mathbf{D} = D_{ii} = B_{im}A_{mi}$.

But $B_{im}A_{mi} = B_{mi}A_{im}$ (change of dummy indices); therefore, we have the desired result

$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}).$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

$\mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j.$

IDENTITY TENSOR

The linear transformation that transforms every vector into itself is called an *identity tensor*. Denoting this special tensor by **I**, we have for any vector **a**,

$$\mathbf{Ia} = \mathbf{a}.$$

In particular,

$$\mathbf{Ie}_1 = \mathbf{e}_1,\ \mathbf{Ie}_2 = \mathbf{e}_2,\ \mathbf{Ie}_3 = \mathbf{e}_3.$$

Thus the (Cartesian) components of the identity tensor are:

$$\mathbf{I}_{ij} = \mathbf{e}_i \cdot \mathbf{Ie}_j = \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij},$$

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

$\mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j.$

that is,

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is obvious that the identity matrix is the matrix of **I** for *all rectangular Cartesian coordinates* and that $\mathbf{TI} = \mathbf{IT} = \mathbf{T}$ for any tensor **T**.

We also note that if $\mathbf{Ta} = \mathbf{a}$ for any arbitrary **a**, then $\mathbf{T} = \mathbf{I}$.

$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j$

$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j.$

$\mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j.$

Example

Write the tensor **T**, defined by the equation $\mathbf{Ta} = \alpha\mathbf{a}$, where α is a constant and **a** is arbitrary, in terms of the identity tensor, and find its components.

We can write $\alpha\mathbf{a}$ as $\alpha\mathbf{Ia}$, so that

$$\mathbf{Ta} = \alpha\mathbf{a} = \alpha\mathbf{Ia}.$$

Since **a** is arbitrary, therefore,

$$\mathbf{T} = \alpha\mathbf{I}.$$

The components of this tensor are clearly $T_{ij} = \alpha\delta_{ij}$.

SCALAR FIELD AND GRADIENT OF A SCALAR FUNCTION

Let $\phi(\mathbf{r})$ be a scalar-valued function of the position vector $\mathbf{r}$. That is, for each position $\mathbf{r}$, $\phi(\mathbf{r})$ gives the value of a scalar, such as density, temperature, or electric potential at the point. In other words, $\phi(\mathbf{r})$ describes a scalar field.

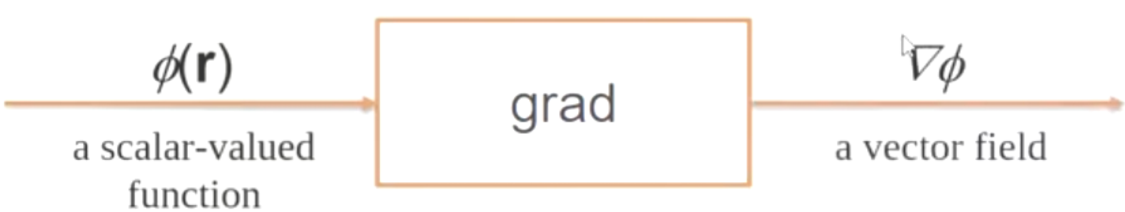
Associated with a scalar field is a vector field, called the *gradient* of ϕ. The gradient of ϕ at a point is defined to be a vector, denoted by grad ϕ or by $\nabla\phi$ such that its dot product with $d\mathbf{r}$ gives the difference of the values of the scalar at $\mathbf{r}+d\mathbf{r}$ and $\mathbf{r}$, i.e.,

$$d\phi = \phi(\mathbf{r}+d\mathbf{r}) - \phi(\mathbf{r}) = \nabla\phi \cdot d\mathbf{r}$$

The Cartesian components of $\nabla\phi$ are $\partial\phi/\partial x_i$, that is,

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i$$

The gradient vector has a simple geometrical interpretation. $\nabla\phi$ is a vector, perpendicular to the surface at the point $\mathbf{r}$.



Example

If $\phi = x_1x_2+2x_3$, find a unit vector **n** normal to the surface of a constant ϕ passing through the point (2,1,0).

Solution:

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 = x_2\mathbf{e}_1 + x_1\mathbf{e}_2 + 2\mathbf{e}_3$$

At the point (2,1,0), $\nabla\phi = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3$. Thus,

$$\mathbf{n} = \frac{1}{3}(\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3)$$

Example

If **q** denotes the heat flux vector (rate of heat transfer/area), the Fourier heat conduction law states that

$$\mathbf{q} = -k\nabla\Theta,$$

where Θ is the temperature field and k is thermal conductivity.

If $\Theta = 2(x_1^2 + x_2^2)$, find $\nabla\Theta$ at the location A (1,0) and $B(1/\sqrt{2}, 1/\sqrt{2})$. Sketch curves of constant Θ (isotherms) and indicate the vectors **q** at the two points.

$$\Theta = 2(x_1^2 + x_2^2)$$
$$\mathbf{q} = -k\nabla\Theta$$
$$A\,(1{,}0) \quad B(1/\sqrt{2}, 1/\sqrt{2})$$

Solution:

$$\nabla\Theta = \frac{\partial\Theta}{\partial x_1}\mathbf{e}_1 + \frac{\partial\Theta}{\partial x_2}\mathbf{e}_2 + \frac{\partial\Theta}{\partial x_3}\mathbf{e}_3 = 4x_1\mathbf{e}_1 + 4x_2\mathbf{e}_2$$

Thus,

$$\mathbf{q} = -4k(x_1\mathbf{e}_1 + x_2\mathbf{e}_2),$$

At point A, $\mathbf{q}_A = -4k\mathbf{e}_1$,
and at point B, $\mathbf{q}_B = -2\sqrt{2}k(\mathbf{e}_1 + \mathbf{e}_2)$

Clearly, the isotherms are circles and the heat flux is an inward radial vector (consistent with heat flowing from higher to lower temperatures).

Example

A more general heat conduction law can be given in the following form:

$$\mathbf{q} = -\mathbf{K}\nabla\Theta$$

where **K** is a tensor known as thermal conductivity tensor. **(a)** What tensor **K** corresponds to the Fourier heat conduction law mentioned in the previous example? **(b)** Find **q** if $\Theta=2x_1+3x_2$, and

$$[\mathbf{K}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution:

(a) Clearly, $\mathbf{K} = k\mathbf{I}$, so that $\mathbf{q} = -kI\nabla\Theta = -k\nabla\Theta$.

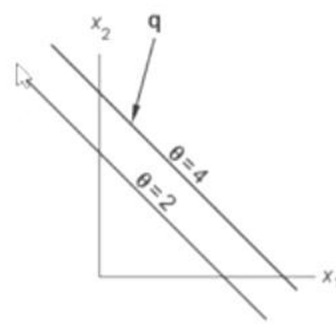
(b) $\nabla\Theta = 2\mathbf{e}_1 + 3\mathbf{e}_2$ and:

$$[\mathbf{q}] = -\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 0 \end{bmatrix}$$

that is,

$$\mathrm{q} = -\mathbf{e}_1 - 4\mathbf{e}_2;$$

which is clearly not normal to the isotherm



VECTOR FIELD AND GRADIENT OF A VECTOR FUNCTION

Let $\mathbf{v}(\mathbf{r})$ be a vector-valued function of position describing, for example, a displacement or a velocity field. Associated with $\mathbf{v}(\mathbf{r})$, is a tensor field, called the *gradient* of $\mathbf{v}$, which is of considerable importance. The gradient of $\mathbf{v}$ (denoted by $\nabla\mathbf{v}$ or grad $\mathbf{v}$) is defined to be the second-order tensor, which, when operating on $d\mathbf{r}$, gives the difference of $\mathbf{v}$ at $\mathbf{r}+d\mathbf{r}$ and $\mathbf{r}$. That is,

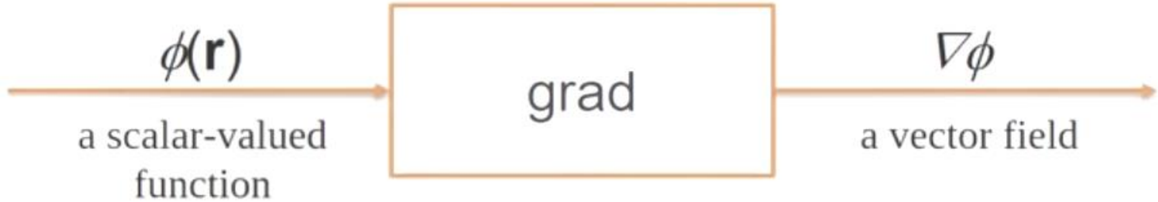
$$d\mathbf{v} = \mathbf{v}(\mathbf{r}+d\mathbf{r}) - \mathbf{v}(\mathbf{r}) = \nabla\mathbf{v}\cdot d\mathbf{r}$$

the components of $\nabla\mathbf{v}$ in indicial notation are given by

$$(\nabla \mathbf{v})_{ij} = \mathbf{e}_i \cdot (\nabla \mathbf{v})\mathbf{e}_j = \mathbf{e}_i \cdot \frac{\partial \mathbf{v}}{\partial x_j} = \frac{\partial (\mathbf{v} \cdot \mathbf{e}_i)}{\partial x_j} = \frac{\partial v_i}{\partial x_j}$$

and in matrix form,

$$[\nabla \mathbf{v}] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$



DIVERGENCE OF A VECTOR FIELD AND DIVERGENCE OF A TENSOR FIELD

Let $\mathbf{v}(\mathbf{r})$ be a vector field. The *divergence* of $\mathbf{v}(\mathbf{r})$ is defined to be a scalar field given by the trace of the gradient of $\mathbf{v}$. That is,

$$\text{div}\,\mathbf{v} \equiv tr(\nabla\mathbf{v})$$

$$[\nabla\mathbf{v}] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

In Cartesian coordinates, this gives

$$\text{div}\,\mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial v_i}{\partial x_i}$$

$$\boxed{\text{div}\,\mathbf{v} = \frac{\partial v_i}{\partial x_i}}$$

Let $\mathbf{T}(\mathbf{r})$ be a tensor field. The divergence of $\mathbf{T}(\mathbf{r})$ is defined to be a vector field, denoted by div $\mathbf{T}$, such that for any vector $\mathbf{a}$

$$(\text{div}\,\mathbf{T})\cdot\mathbf{a} \equiv \text{div}(\mathbf{T}^{\mathrm{T}}\mathbf{a}) - tr(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{a}).$$

To find the Cartesian components of the vector div $\mathbf{T}$, let $\mathbf{b}$ = div $\mathbf{T}$, then (Note: $\nabla\mathbf{e}_i = \mathbf{0}$ for Cartesian coordinates), from the last equation, we have

$$b_i = \mathbf{b}\cdot\mathbf{e}_i = \text{div}\,\mathbf{T}\cdot\mathbf{e}_i = \text{div}(\mathbf{T}^{\mathrm{T}}\mathbf{e}_i) - tr(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{e}_i) = \text{div}(T_{ji}\mathbf{e}_i) - 0$$
$$= \text{div}(T_{ij}\mathbf{e}_j) = \partial T_{ij}/\partial x_j$$

In other words,

$$\text{div}\,\mathbf{T} = \left(\frac{\partial T_{ij}}{\partial x_j}\right)\mathbf{e}_i$$

Example

Let $\alpha = \alpha(\mathbf{r})$ and $\mathbf{a}$=$\mathbf{a}(\mathbf{r})$. Show that $\operatorname{div}(\alpha\mathbf{a}) = \alpha \operatorname{div}\mathbf{a} + (\nabla\alpha)\cdot\mathbf{a}$

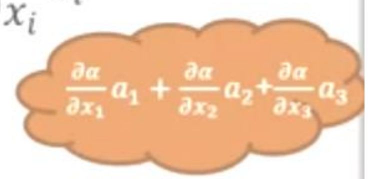
Solution:

Let $\mathbf{b} = \alpha\mathbf{a}$. Then $b_i = \alpha a_i$, so

$$\operatorname{div}\mathbf{b} = \frac{\partial b_i}{\partial x_i} = \frac{\partial(\alpha a_i)}{\partial x_i} = \alpha\frac{\partial a_i}{\partial x_i} + \frac{\partial \alpha}{\partial x_i}a_i$$

That is,

$$\operatorname{div}(\alpha\mathbf{a}) = \alpha \operatorname{div}\mathbf{a} + (\nabla\alpha)\cdot\mathbf{a}$$



$\operatorname{div} \mathbf{T} = \left(\frac{\partial T_{ij}}{\partial x_j}\right)\mathbf{e}_i$

Example

Let $\alpha = \alpha(\mathbf{r})$ and $\mathbf{T}$=$\mathbf{T}(\mathbf{r})$. Show that $\operatorname{div}(\alpha\mathbf{T}) = \mathbf{T}(\nabla\alpha) + \alpha \operatorname{div} \mathbf{T}$

Solution:

We have

$$\operatorname{div}(\alpha\mathbf{T}) = \frac{\partial(\alpha T_{ij})}{\partial x_j}\mathbf{e}_i = \frac{\partial \alpha}{\partial x_j} T_{ij}\mathbf{e}_i + \alpha\frac{\partial T_{ij}}{\partial x_j}\mathbf{e}_i$$

$$= \mathbf{T}(\nabla\alpha) + \alpha \operatorname{div} \mathbf{T}$$

LAPLACIAN OF A SCALAR FIELD

Let $f(\mathbf{r})$ be a scalar-valued function of the position vector $\mathbf{r}$. The definition of the Laplacian of a scalar field is given by

$$\nabla^2 f = \operatorname{div}(\nabla f) = tr\big(\nabla(\nabla f)\big).$$

In rectangular coordinates the Laplacian becomes

$$\nabla^2 f = tr\big(\nabla(\nabla f)\big) = \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

Problems

(4) Consider the scalar field $\phi = x_1^2 + 3x_1x_2 + 2x_3$,

(a) Find the unit vector normal to the surface of constant ϕ at the origin and at (1,0,1).

(b) What is the maximum value of the directional derivative of ϕ at the origin? at (1,0,1)?

Problems

(5) Consider the ellipsoidal surface defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Find the unit vector normal to the surface at a given point (x, y, z).

Problems

(6) Consider the temperature field given by $\Theta = 3x_1x_2$.

(a)If $\mathbf{q} = -k\nabla\Theta$, find the heat flux at the point $A(1,1,1)$.

(b)If $\mathbf{q} = -\mathbf{K}\nabla\Theta$, find the heat flux at the same point, where

$$[\mathbf{K}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 3k \end{bmatrix}$$

KINEMATICS OF A CONTINUUM

In continuum mechanics, materials are generally classified as **solid** and **fluids**, depending on their behavior when subjected to loading.

We consider that a **solid body deforms** when subjected to external forces while **fluid body flows**.

The study of geometric changes in a continuum without regard to the force causing the change is known as *kinematics*.

DESCRIPTION OF MOTIONS OF A CONTINUUM

In particle kinematics, the path line of a particle is described by a vector function of time t, $\mathbf{r}=\mathbf{r}(t)$, where $\mathbf{r}(t)=x_1(t)\mathbf{e}_1+x_2(\mathrm{t})\mathbf{e}_2+x_3(t)\mathbf{e}_3$ is the position vector. In component form, the previous equation reads:

$$x_1=x_1(t),\ x_2=x_2(t),\ x_3=x_3(t)$$

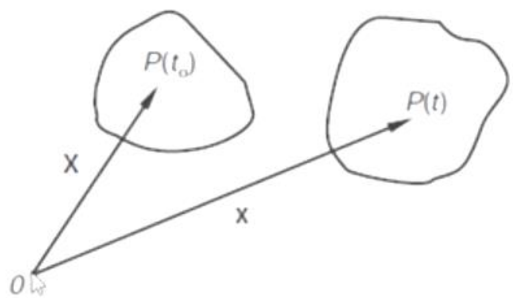
If there are N particles, there are N path lines, each of which is described by one of the equations:

$$\mathbf{r}_\mathrm{n}=\mathbf{r}_\mathrm{n}(t),\ \mathrm{n}=1,\ 2,\ 3\ldots,\ N$$

That is, for the particle number 1, the path line is given by $\mathbf{r}_1(t)$, for the particle number 2, it is given by $\mathbf{r}_2(t)$, etc.

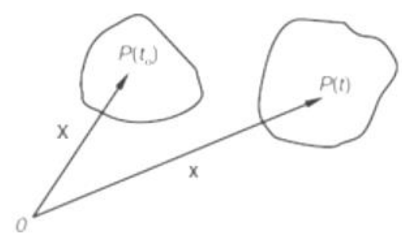
DESCRIPTION OF MOTIONS OF A CONTINUUM

For a continuum, there are infinitely many particles. Therefore, it is not possible to identify particles by assigning each of them a number in the same way as in the kinematics of particles. However, it is possible to identify them by the position they occupy at some reference time t_o.



DESCRIPTION OF MOTIONS OF A CONTINUUM

If a particle of a continuum was at the position (X_1,X_2,X_3) at the reference time t_o, the set of coordinates (X_1,X_2,X_3) can be used to identity this particle.



Thus, in general, the path lines of every particle in a continuum can be described by a vector equation of the form

$$\mathbf{x}=\mathbf{x}(\mathbf{X},t) \quad \text{with} \quad \mathbf{X}=\mathbf{x}(\mathbf{X},t_0),$$

where $\mathbf{x}=x_1\mathbf{e}_1+x_2\mathbf{e}_2+x_3\mathbf{e}_3$ is the position vector at time t for the particle P, which was at $\mathbf{X}=X_1\mathbf{e}_1+X_2\mathbf{e}_2+X_3\mathbf{e}_3$ at time t_0.

DESCRIPTION OF MOTIONS OF A CONTINUUM

In component form

$$x_1 = x_1(X_1, X_2, X_3, t), \qquad X_1 = x_1(X_1, X_2, X_3, t_0),$$
$$x_2 = x_2(X_1, X_2, X_3, t), \qquad X_2 = x_2(X_1, X_2, X_3, t_0),$$
$$x_3 = x_3(X_1, X_2, X_3, t), \qquad X_3 = x_3(X_1, X_2, X_3, t_0),$$

or

$$x_i = x_i(X_1, X_2, X_3, t), \qquad X_i = x_i(X_1, X_2, X_3, t_0),$$

In the first set of equations, the triple (X_1,X_2,X_3) serves to identify the different particles of the body and is known as the *material coordinates*. While both equation above is said to define a *motion* for a continuum; these equations describe the *path line* for every *particle* in the continuum.

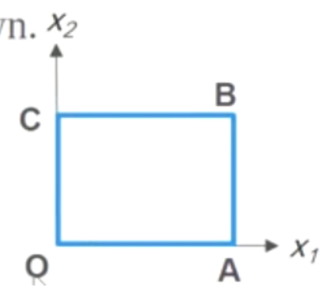
Example 1

Consider the motion

$$\mathbf{x} = \mathbf{X} + ktX_2\mathbf{e}_1$$

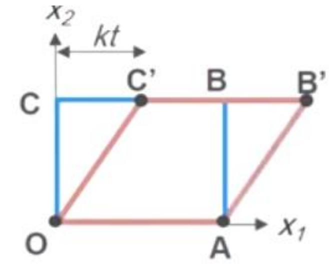
where $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ is the position vector at time t for a particle P that was at $\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2$ at t=0.

Sketch the configuration at time t for a body which, at t=0, has the shape of a square of unit sides as shown.



Since $\mathbf{x} = \mathbf{X} + ktX_2\mathbf{e}_1$, then

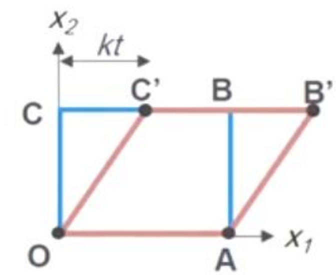
$$x_1 = X_1 + ktX_2, \qquad x_2 = X_2, \qquad x_3 = X_3$$



#	X	x	Comments
1	**O**: (0,0,0)	**O**: (0,0,0)	No Change
2	**A**: (1,0,0)	**A**: (1,0,0)	No Change
3	**OA**: (X_1,0,0)	**OA**: (X_1,0,0)	No Change
4	**CB**: (X_1,1,0)	**C'B'**: (X_1+kt,1,0)	The material line has moved horizontally through a distance of kt.
5	**OC**: (0,X_2,0)	**OC'**: (ktX_2,X_2,0)	The fact that $x_1=ktX_2$ means that the straight material line **OC** remains straight line **OC'** at time t
6	**AB**: (1,X_2,0)	**AB'**: (1+ktX_2,X_2,0)	Similar to **OC**.

Since $\mathbf{x} = \mathbf{X} + ktX_2\mathbf{e}_1$, then

$$x_1 = X_1 + ktX_2, \qquad x_2 = X_2, \qquad x_3 = X_3$$



Thus, at time t, the side of the square change from the square to a parallelogram.

Since $x_3 = X_3$ at all time for all particles, it is clear that all motions are parallel to the plane $x_3 = 0$. The motion given in this example is known as the *simple shearing motion*.

MATERIAL DESCRIPTION AND SPATIAL DESCRIPTION

When a continuum is in motion, its temperature $\mathbf{\Theta}$, its velocity $\mathbf{v}$, and its stress tensor $\mathbf{T}$ may change with time. We can describe these changes as follows.

1. Following the particles, i.e., we express $\mathbf{\Theta}$, $\mathbf{v}$, $\mathbf{T}$ as functions of the particles [identified by the material coordinates (X_1, X_2, X_3, t)] and time t. In other words, we express

$$\mathbf{\Theta} = \hat{\mathbf{\Theta}}(X_1, X_2, X_3, t)$$
$$\mathbf{v} = \hat{\mathbf{v}}(X_1, X_2, X_3, t)$$
$$\mathbf{T} = \hat{\mathbf{T}}(X_1, X_2, X_3, t)$$

Such a description is known as the *material description*. Other names for it are the *Lagrangean description* and the *reference description*.

MATERIAL DESCRIPTION AND SPATIAL DESCRIPTION

When a continuum is in motion, its temperature $\mathbf{\Theta}$, its velocity $\mathbf{v}$, and its stress tensor $\mathbf{T}$ may change with time. We can describe these changes as follows.

2. Observing the changes at fixed locations, i.e., we express $\mathbf{\Theta}$, $\mathbf{v}$, $\mathbf{T}$ as functions of fixed position and time. Thus,

$$\mathbf{\Theta} = \check{\mathbf{\Theta}}(x_1, x_2, x_3, t)$$
$$\mathbf{v} = \check{\mathbf{v}}(x_1, x_2, x_3, t)$$
$$\mathbf{T} = \check{\mathbf{T}}(x_1, x_2, x_3, t)$$

Such a description is known as a *spatial description* or *Eulerian description*.

Example 2

Given the motion of continuum to be

$$x_1 = X_1 + ktX_2, \qquad x_2 = (1 + kt)X_2, \qquad x_3 = X_3. \tag{i}$$

If the temperature field is given by the spatial description

$$\boldsymbol{\Theta} = \alpha(x_1 + x_2). \tag{ii}$$

(a) Find the material description of temperature and

(b) Obtain the velocity and the rate of change of the temperature for particular material particles and express the answer in both a material and a spatial description.

Solution:

(a) We have

$$x_1 = X_1 + ktX_2, \qquad x_2 = (1 + kt)X_2, \qquad x_3 = X_3. \tag{i}$$

$$\boldsymbol{\Theta} = \alpha(x_1 + x_2). \tag{ii}$$

Substituting Eq. (i) into Eq. (ii), we obtain the material description for the temperature,

$$\boldsymbol{\Theta} = \alpha(x_1 + x_2) = \alpha X_1 + \alpha(1 + 2kt)X_2. \tag{iii}$$

(b) Since a particular material is designated by a specific X, its velocity will be given by

$$v_i = \left(\frac{\partial x_i}{\partial t}\right)_{X_i-fixed} \tag{iv}$$

$$x_1 = X_1 + ktX_2, \qquad x_2 = (1 + kt)X_2, \qquad x_3 = X_3. \quad \text{(i)}$$

$$v_i = \left(\frac{\partial x_i}{\partial t}\right)_{X_i-fixed} \quad \text{(iv)}$$

So that from Eq. (i)

$$v_1 = kX_2, \quad v_2 = kX_2, \qquad v_3 = 0. \quad \text{(v)}$$

This is material description of the velocity field.
To obtain the spatial description, we make use Eq. (i) again, where we have

$$X_2 = \frac{x_2}{(1+kt)} \quad \text{(vi)}$$

Therefore, the spatial decryption for the velocity field is

$$v_1 = \frac{kx_2}{(1+kt)}, \quad v_2 = \frac{kx_2}{(1+kt)}, \qquad v_3 = 0. \quad \text{(vii)}$$

$$\boldsymbol{\Theta} = \alpha(x_1 + x_2) = \alpha X_1 + \alpha(1 + 2kt)X_2. \quad \text{(iii)}$$

$$X_2 = \frac{x_2}{(1+kt)} \quad \text{(vi)}$$

From Eq. (iii), in material description, the rate of change of temperature for particular material particles is given by

$$\left(\frac{\partial \boldsymbol{\Theta}}{\partial t}\right)_{X_i-fixed} = 2\alpha k X_2 \quad \text{(viii)}$$

To obtain the spatial description, we substitute Eq. (vi) in Eq. (viii):

$$\left(\frac{\partial \boldsymbol{\Theta}}{\partial t}\right)_{X_i-fixed} = \frac{2\alpha k x_2}{(1 + kt)}.$$

Example 3

The position at time t of a particle initially at (X_1, X_2, X_3) is given by the equations

$$x_1 = X_1 + k(X_1 + X_2)t, \quad x_2 = X_2 + k(X_1 + X_2)t, \quad x_3 = X_3 \qquad \text{(i)}$$

(a) Find the velocity at t=2 for the particle that was at (1,1,0) at the reference time.

(b) Find the velocity at t=2 for the particle that is at the position (1,1,0) at t=2.

Solution:

$$x_1 = X_1 + k(X_1 + X_2)t, \quad x_2 = X_2 + k(X_1 + X_2)t, \quad x_3 = X_3 \qquad \text{(i)}$$

(a) We have $x_1 = X_1 + k(X_1 + X_2)t, \quad x_2 = X_2 + k(X_1 + X_2)t, \quad x_3 = X_3$

$$v_1 = \left(\frac{\partial x_1}{\partial t}\right)_{X_i-fixed} = k(X_1 + X_2), v_2 = \left(\frac{\partial x_2}{\partial t}\right)_{X_i-fixed} = k(X_1 + X_2), v_3 = 0 \quad \text{(ii)}$$

For the particle (X_1,X_2,X_3)=(1,1,0), the velocity at t=2 is

$$v_1 = k(1+1) = 2k, \qquad v_2 = k(1+1) = 2k, \qquad v_3 = 0$$

that is,

$$\mathbf{v} = 2k\mathbf{e}_1 + 2k\mathbf{e}_2.$$

(b) We need to calculate the reference position at (X_1, X_2, X_3) that was occupied by the particle which, at t=2, is at (x_1, x_2, x_3)=(1,1,0). To do this, we substitute this condition into Eq. (i) and solve for at (X_1, X_2, X_3) that is,

$$1 = (1+2k)X_1 + 2kX_2, \qquad 1 = (1+2k)X_2 + 2kX_1$$

$$1 = (1 + 2k)X_1 + 2kX_2, \qquad 1 = (1 + 2k)X_2 + 2kX_1$$

$$\begin{bmatrix} 1 + 2k & 2k \\ 2k & 1 + 2k \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

thus,

$$X_1 = \frac{1}{1 + 4k}, \qquad X_2 = \frac{1}{1 + 4k}.$$

Substituting these values in Eq. (ii),

$$v_1 = \left(\frac{\partial x_1}{\partial t}\right)_{X_i - fixed} = k(X_1 + X_2), v_1 = \left(\frac{\partial x_2}{\partial t}\right)_{X_i - fixed} = k(X_1 + X_2), v_3 = 0 \quad \text{(ii)}$$

we obtain

$$v_1 = \frac{2k}{1 + 4k}, \qquad v_2 = \frac{2k}{1 + 4k}, \qquad v_3 = 0$$

MATERIAL DERIVATIVE

The time rate of change of a quantity (such as temperature or velocity or stress tensor) of a material particle is known as a *material derivative*. We shall denote the material derivative by *D/Dt:*

1. When a material description of a scalar quantity is used, we have

$$\boldsymbol{\Theta} = \widehat{\boldsymbol{\Theta}}(X_1, X_2, X_3, t)$$

then

$$\boxed{\frac{D\boldsymbol{\Theta}}{Dt} = \left(\frac{\partial \widehat{\boldsymbol{\Theta}}}{\partial t}\right)_{X_i-fixed}}$$

MATERIAL DERIVATIVE

2. When a spatial description of the same quantity is used, we have

$$\boldsymbol{\Theta} = \breve{\boldsymbol{\Theta}}\,(x_1, x_2, x_3, t)$$

where x_j, the coordinates of the present positions of material particles at time t are related to material coordinates by the known motion $x_i = \hat{x}_i(X_1, X_2, X_3, t)$

$$\frac{D\boldsymbol{\Theta}}{Dt} = \left(\frac{\partial \widehat{\boldsymbol{\Theta}}}{\partial t}\right)_{X_i-fixed} = \left(\frac{\partial \breve{\boldsymbol{\Theta}}}{\partial x_1}\right)\frac{\partial \hat{x}_1}{\partial t} + \left(\frac{\partial \breve{\boldsymbol{\Theta}}}{\partial x_2}\right)\frac{\partial \hat{x}_2}{\partial t} + \left(\frac{\partial \breve{\boldsymbol{\Theta}}}{\partial x_3}\right)\frac{\partial \hat{x}_3}{\partial t} + \left(\frac{\partial \breve{\boldsymbol{\Theta}}}{\partial t}\right)_{X_i-fixed}$$

Where $\frac{\partial \hat{x}_1}{\partial t}, \frac{\partial \hat{x}_2}{\partial t}$, and $\frac{\partial \hat{x}_3}{\partial t}$ are to be obtained with fixed values of the X_i's. When rectangular Cartesian coordinates are used, these are the velocity components v_i of the particle X_i. Thus, the material derivative in rectangular coordinates is

$$\frac{D\boldsymbol{\Theta}}{Dt} = \left(\frac{\partial \widehat{\boldsymbol{\Theta}}}{\partial t}\right)_{X_i-fixed} = \frac{\partial \breve{\boldsymbol{\Theta}}}{\partial t} + v_1\left(\frac{\partial \breve{\boldsymbol{\Theta}}}{\partial x_1}\right) + v_2\left(\frac{\partial \breve{\boldsymbol{\Theta}}}{\partial x_2}\right) + v_3\left(\frac{\partial \breve{\boldsymbol{\Theta}}}{\partial x_3}\right)$$

MATERIAL DERIVATIVE

$$\frac{D\Theta}{Dt} = \left(\frac{\partial\hat{\Theta}}{\partial t}\right)_{X_i-fixed} = \frac{\partial\breve{\Theta}}{\partial t} + v_1\left(\frac{\partial\breve{\Theta}}{\partial x_1}\right) + v_2\left(\frac{\partial\breve{\Theta}}{\partial x_2}\right) + v_3\left(\frac{\partial\breve{\Theta}}{\partial x_3}\right)$$

or, in indicial notation,

$$\frac{D\Theta}{Dt} = \left(\frac{\partial\hat{\Theta}}{\partial t}\right)_{X_i-fixed} = \frac{\partial\breve{\Theta}}{\partial t} + v_i\left(\frac{\partial\breve{\Theta}}{\partial x_i}\right)$$

and in direct notation,

$$\boxed{\frac{D\Theta}{Dt} = \frac{\partial\breve{\Theta}}{\partial t} + \mathbf{v}\cdot\nabla\breve{\Theta}}$$

Example

Obtain $D\Theta/Dt$ for the motion given by

$$x_1 = X_1 + ktX_2, \qquad x_2 = (1+kt)\,X_2, \qquad x_3 = X_3. \qquad \text{(i)}$$

and the temperature field is given by the spatial description

$$\Theta = \alpha(x_1 + x_2). \qquad \text{(ii)}$$

Solution:

$$\frac{D\Theta}{Dt} = \frac{\partial\breve{\Theta}}{\partial t} + \mathbf{v}\cdot\nabla\breve{\Theta}$$

From Example 2 (in previous lecture), we have

$$\mathbf{v} = \frac{kx_2}{(1+kt)}(\mathbf{e}_1 + \mathbf{e}_2) \quad \text{and } \Theta = \alpha(x_1 + x_2).$$

The gradient of Θ is simply $\alpha(\mathbf{e}_1 + \mathbf{e}_2)$, therefore,

$$\frac{D\Theta}{Dt} = \frac{\partial\breve{\Theta}}{\partial t} + \mathbf{v}\cdot\nabla\breve{\Theta}$$

$$\frac{D\Theta}{Dt} = 0 + \frac{kx_2}{(1+kt)}(\mathbf{e}_1 + \mathbf{e}_2)\cdot\alpha(\mathbf{e}_1 + \mathbf{e}_2) = \frac{2\alpha kx_2}{(1+kt)}$$

ACCELERATION OF A PARTICLE

The acceleration of a particle is the rate of change of velocity of the particle. It is, therefore, the material derivative of velocity. If the motion of a continuum is given by,

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \quad \text{with} \quad \mathbf{X} = \mathbf{x}(\mathbf{X}, t_0)$$

Then the velocity **v** at time *t* of a particle **X** is given by

$$\mathbf{v} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{X_i-fixed} \equiv \frac{D\mathbf{x}}{Dt}$$

And the acceleration **a** at time *t* of a particle **X** is given by

$$\mathbf{a} = \left(\frac{\partial \mathbf{v}}{\partial t}\right)_{X_i-fixed} \equiv \frac{D\mathbf{v}}{Dt}$$

ACCELERATION OF A PARTICLE

Thus, if the material description of velocity $\mathbf{v}(\mathbf{X},t)$ is known, then the acceleration is very easily computed, simply taking the partial derivative with respect to time of the function.

On the other hand, if only the spatial description of velocity [i.e., $\mathbf{v}=\mathbf{v}(\mathbf{x},t)$] is known, the computation of acceleration is not as simple. We derive the formulas for its computation in the following:

Rectangular Cartesian coordinates (x_1,x_2,x_3). With

$$\mathbf{v} = v_1(x_1, x_2, x_3, t)e_1 + v_2(x_1, x_2, x_3, t)e_2 + v_3(x_1, x_2, x_3, t)e_3$$

we have, since the base vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are fixed vectors,

$$\mathbf{a} = \frac{D\mathbf{v}}{Dt} = \frac{Dv_1}{Dt}\mathbf{e}_1 + \frac{Dv_2}{Dt}\mathbf{e}_2 + \frac{Dv_3}{Dt}\mathbf{e}_3$$

ACCELERATION OF A PARTICLE

In component form, we have

$$a_i = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_1 \frac{\partial v_i}{\partial x_1} + v_2 \frac{\partial v_i}{\partial x_2} + v_3 \frac{\partial v_i}{\partial x_3}$$

or

$$a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}$$

In a form valid for all coordinates systems, we have

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v})\mathbf{v}$$

Example

Given the velocity field

$$v_1 = \frac{kx_1}{1+kt}, \quad v_2 = \frac{kx_2}{1+kt}, \quad v_3 = \frac{kx_3}{1+kt}$$

(a) Find the acceleration field and **(b)** Find the path line $\mathbf{x} = \hat{\mathbf{x}}(\mathbf{X}, t)$.

Solution: (a) With

$$v_i = \frac{kx_i}{1+kt}$$

we have

$$a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{k^2 x_i}{(1+kt)^2} + \frac{kx_j}{1+kt}\frac{k\delta_{ij}}{1+kt} = -\frac{k^2 x_i}{(1+kt)^2} + \frac{kx_i}{(1+kt)^2} = 0$$

OR

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v})\mathbf{v}$$

$$v_1 = \frac{kx_1}{1+kt}, \quad v_2 = \frac{kx_2}{1+kt}, \quad v_3 = \frac{kx_3}{1+kt}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial v_1}{\partial t} \\ \frac{\partial v_2}{\partial t} \\ \frac{\partial v_3}{\partial t} \end{bmatrix} + \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -\frac{k^2 x_1}{(1+kt)^2} \\ -\frac{k^2 x_2}{(1+kt)^2} \\ -\frac{k^2 x_3}{(1+kt)^2} \end{bmatrix} + \begin{bmatrix} \frac{k}{1+kt} & 0 & 0 \\ 0 & \frac{k}{1+kt} & 0 \\ 0 & 0 & \frac{k}{1+kt} \end{bmatrix} \begin{bmatrix} \frac{kx_1}{1+kt} \\ \frac{kx_2}{1+kt} \\ \frac{kx_3}{1+kt} \end{bmatrix} = \begin{bmatrix} -\frac{k^2 x_1}{(1+kt)^2} \\ -\frac{k^2 x_2}{(1+kt)^2} \\ -\frac{k^2 x_3}{(1+kt)^2} \end{bmatrix} + \begin{bmatrix} \frac{k^2 x_1}{(1+kt)^2} \\ \frac{k^2 x_2}{(1+kt)^2} \\ \frac{k^2 x_3}{(1+kt)^2} \end{bmatrix} = 0$$

(b) Find the path line $\mathbf{x} = \hat{\mathbf{x}}(\mathbf{X}, t)$. Since

$$v_i = \left(\frac{\partial x_i}{\partial t}\right)_{X_i - fixed} = \frac{kx_i}{1+kt}$$

therefore,

$$\int_{X_1}^{x_1} \frac{dx_1}{kx_1} = \int_0^t \frac{dt}{1+kt}$$

that is,

$$\frac{1}{k}(lnx_1 - lnX_1) = \frac{1}{k}\ln(1+kt) \Rightarrow ln\left(\frac{x_1}{X_1}\right) = ln(1+kt)$$

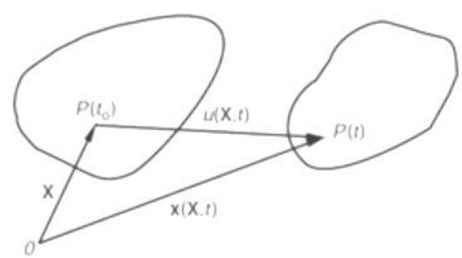
$$\Rightarrow x_1 = (1+kt)\,X_1$$

Similarly,

$$x_2 = (1+kt)\,X_2$$
$$x_3 = (1+kt)\,X_3$$

DISPLACEMENT FIELD

The displacement vector of a particle in a continuum (identified by its material coordinate X), from the reference position $P(t_0)$ to the current position $P(t)$ is given by the vector from $P(t_0)$ to $P(t)$ and is denoted by $\mathbf{u}(\mathbf{X},t)$



That is,

$$\mathbf{u}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X}$$

From the preceding equation, it is clear that whenever the path lines of a continuum are known, its displacement field is also known.

Example

The position at time t of a particle initially at (X_1,X_2,X_3) is given by

$$x_1 = X_1 + (X_1 + X_2)kt, \quad x_2 = X_2 + (X_1 + X_2)kt, \quad x_3 = X_3$$

obtain the displacement field.

Solution:

$$u_1 = x_1 - X_1 = [X_1 + (X_1 + X_2)kt] - X_1 = (X_1 + X_2)kt,$$
$$u_2 = x_2 - X_2 = [X_2 + (X_1 + X_2)kt] - X_2 = (X_1 + X_2)kt,$$
$$u_3 = x_3 - X_3 = X_3 - X_3 = 0.$$

Example

The deformed configuration of a continuum is given by

$$x_1 = \frac{1}{2}X_1, \quad x_2 = X_2, \quad x_3 = X_3$$

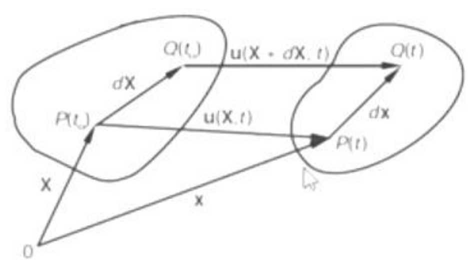
obtain the displacement field.

Solution:

$$u_1 = x_1 - X_1 = \frac{1}{2}X_1 - X_1 = -\frac{1}{2}X_1,$$
$$u_2 = x_2 - X_2 = X_2 - X_2 = 0,$$
$$u_3 = x_3 - X_3 = X_3 - X_3 = 0.$$

INFINITESIMAL DEFORMATION

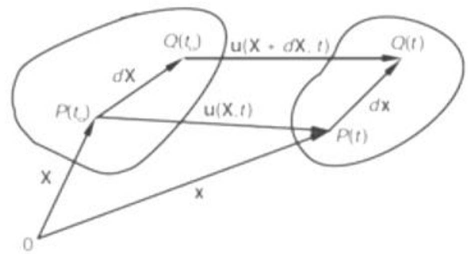
There are many important engineering problems that involve structural members or machine parts for which the deformation is very small (mathematically treated as infinitesimal). In this section, we derive the tensor that characterizes the deformation of such bodies.



Consider a body having a particular configuration at some reference time t_o, changes to another configuration at time t. Referring to Figure above, a typical material point P undergoes a displacement **u** so that it arrives at the position

INFINITESIMAL DEFORMATION

$$\mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X}, t) \qquad (1)$$



A neighboring point Q at $\mathbf{X}+d\mathbf{X}$ arrives at $\mathbf{x}+d\mathbf{x}$, which is related to $\mathbf{X}+d\mathbf{X}$ by

$$\mathbf{x} + d\mathbf{x} = \mathbf{X} + d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}, t) \qquad (2)$$

Subtracting Eq. (1) from Eq. (2), we obtain

$$d\mathbf{x} = d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}, t) - \mathbf{u}(\mathbf{X}, t) \qquad (3)$$

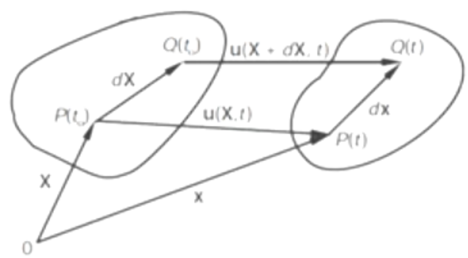
Using the definition of gradient of a vector function $[d\mathbf{v} = \mathbf{v}(\mathbf{r} + d\mathbf{r}) - \mathbf{v}(\mathbf{r}) = (\nabla\mathbf{v})d\mathbf{r}]$, Eq. (3) becomes

$$d\mathbf{x} = d\mathbf{X} + (\nabla\mathbf{u})d\mathbf{X}$$

where $\nabla\mathbf{u}$ is a second-order tensor known as the *displacement gradient*.

INFINITESIMAL DEFORMATION

The matrix of $\nabla\mathbf{u}$ with respect to rectangular Cartesian coordinates ($\mathbf{X}=X_i\mathbf{e}_i$ and $\mathbf{u}=u_i\mathbf{e}_i$) is



$$[\nabla\mathbf{u}] = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix}$$

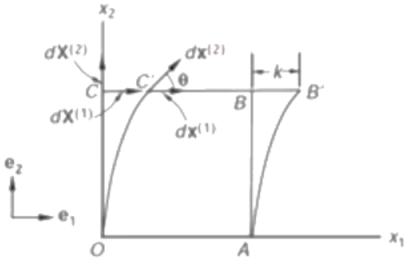
Example

Given the following displacement components $u_1 = kX_2^2$, $u_2 = u_3 = 0$.

(a) Sketch the deformed shape of the unit square OABC shown in the Figure below.

(b) Find the deformed vectors (i.e., $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ of the material elements $d\mathbf{X}^{(1)}=d\mathbf{X}_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)}=d\mathbf{X}_2\mathbf{e}_2$, which were at the point C.

(c) Determine the ratio of the deformed to the undeformed lengths of the differential elements (known as stretch) of part **(b)** and the change in angle between these elements.



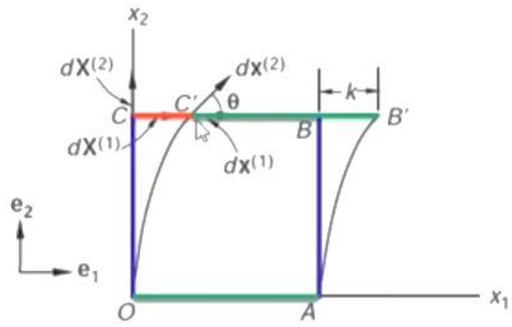
(a) Sketch the deformed shape of the unit square OABC shown in the Figure below.

Solution:

For the material line OA

$X_2=0$, therefore, $u_1=u_2=u_3=0$.

That is, the line is not displaced.



For the material line CB

$X_2=1$, therefore, $u_1=k$, $u_2=u_3=0$.

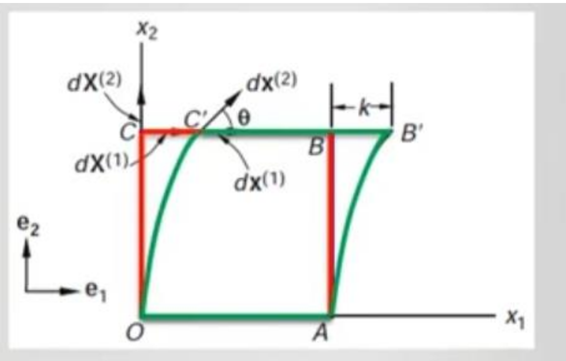
the line is displaced by k units to the right

For the material line OC and AB,

$u_1 = kX_2^2$, $u_2 = u_3 = 0$.

each line becomes parabolic in shape.

Thus, the deformed shape is given by OAB'C'

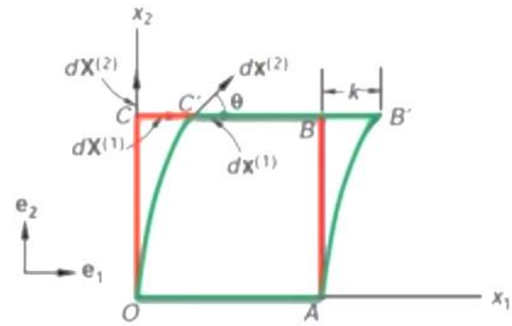


(b) Find the deformed vectors (i.e., $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ of the material elements $d\mathbf{X}^{(1)}=d\mathbf{X}_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)}=d\mathbf{X}_2\mathbf{e}_2$, which were at the point C.

Solution:

For the material point C, the matrix of the displacement gradient is

$$[\nabla \mathbf{u}] = \begin{bmatrix} 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{X_2=1} = \begin{bmatrix} 0 & 2k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Therefore, for $d\mathbf{X}^{(1)}=dX_1\mathbf{e}_1$, from the Eq.

$$d\mathbf{x} = d\mathbf{X} + (\nabla \mathbf{u})d\mathbf{X}$$

$$d\mathbf{x}^{(1)} = d\mathbf{X}^{(1)} + (\nabla \mathbf{u})d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1 + 0 = dX_1\mathbf{e}_1$$

and for $d\mathbf{X}^{(2)}=dX_2\mathbf{e}_2$

$$d\mathbf{x}^{(2)} = d\mathbf{X}^{(2)} + (\nabla \mathbf{u})d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2 + 2kdX_2\mathbf{e}_1 = dX_2(2k\mathbf{e}_1 + \mathbf{e}_2)$$

(c) Determine the ratio of the deformed to the undeformed lengths of the differential elements (known as stretch) of part **(b)** and the change in angle between these elements.

Solution:

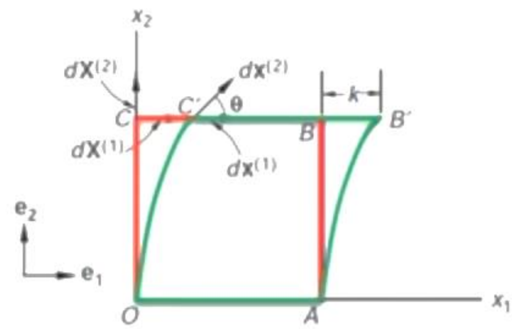
From part (b), we have

$$d\mathbf{x}^{(1)} = dX_1\mathbf{e}_1$$

$$d\mathbf{x}^{(2)} = dX_2(2k\mathbf{e}_1 + \mathbf{e}_2)$$

$$|d\mathbf{x}^{(1)}| = dX_1$$

$$|d\mathbf{x}^{(2)}| = dX_2\sqrt{4k^2+1}$$

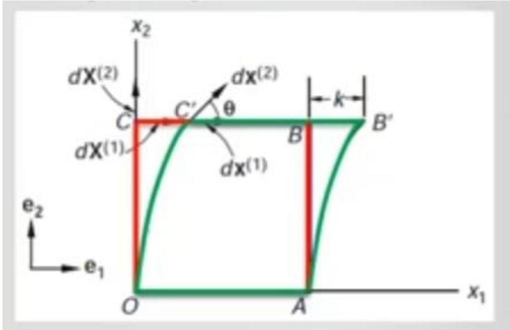


therefore,

$$\frac{|d\mathbf{x}^{(1)}|}{|d\mathbf{X}^{(1)}|}=1 \quad \text{and} \quad \frac{|d\mathbf{x}^{(2)}|}{|d\mathbf{X}^{(2)}|}=\sqrt{4k^2+1}$$

$d\mathbf{X}^{(1)}=d\mathbf{X}_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)}=d\mathbf{X}_2\mathbf{e}_2$

$$\cos\theta = \frac{d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}}{|d\mathbf{x}^{(1)}||d\mathbf{x}^{(2)}|} = \frac{2k}{\sqrt{1+4k^2}}$$



If k is very small, we have the case of small deformations, and by the binomial theorem, we keep only the first power of k,

$$\frac{|d\mathbf{x}^{(1)}|}{|d\mathbf{X}^{(1)}|} = 1 \quad \text{and} \quad \frac{|d\mathbf{x}^{(2)}|}{|d\mathbf{X}^{(2)}|} = \sqrt{1+4k^2} \approx 1 + \frac{1}{2}(4k^2)$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$(1+x)^n \approx 1 + nx$$

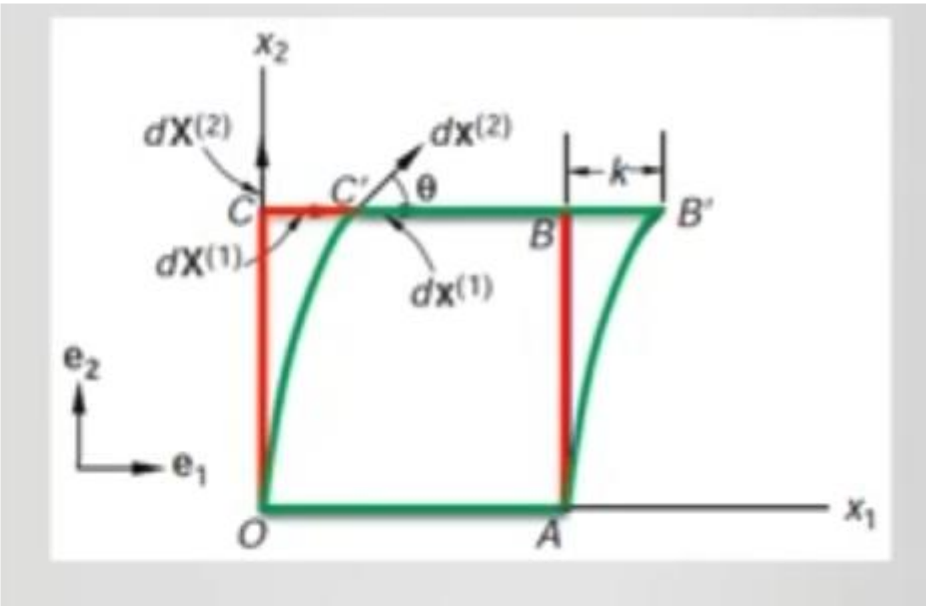
and $\cos\theta \approx 2k$

If γ denotes the decrease in angle, then

$$\cos\theta = \cos\left(\frac{\pi}{2} - \gamma\right) = \sin\gamma = 2k$$

Now, for very small k, γ is also small, so that $\sin\gamma \approx \gamma$ and we have

$$\gamma \approx 2k$$



STRAIN TENSOR

We know that $d\mathbf{x} = d\mathbf{X} + (\nabla\mathbf{u})d\mathbf{X}$

$$\Rightarrow d\mathbf{x} = (\mathbf{I} + \nabla\mathbf{u})d\mathbf{X} \qquad \Rightarrow d\mathbf{x} = \mathbf{F}d\mathbf{X} \tag{1}$$

where $\mathbf{F} = (\mathbf{I} + \nabla\mathbf{u})$, and it is called the *deformation gradient*.

To find the relationship between ds (the length of $d\mathbf{x}$) and dS (the length of $d\mathbf{X}$), we take the dot product of Eq. (1) with itself:

$$cos\theta = \frac{d\mathbf{x}\cdot d\mathbf{x}}{|d\mathbf{x}||d\mathbf{x}|}$$

$$\theta = 0 \Rightarrow 1 = \frac{d\mathbf{x}\cdot d\mathbf{x}}{ds\,ds}$$

$$d\mathbf{x}\cdot d\mathbf{x} = ds^2$$

$$d\mathbf{x}\cdot d\mathbf{x} = \underbrace{\boxed{\mathbf{F}d\mathbf{X}}}_{\mathbf{a}} \cdot \mathbf{F}\underbrace{\boxed{d\mathbf{X}}}_{\mathbf{b}}$$

$$\mathbf{a}\cdot\mathbf{T}\mathbf{b} = \mathbf{b}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{a}$$

$$d\mathbf{x}\cdot d\mathbf{x} = d\mathbf{X}\cdot\mathbf{F}^{\mathrm{T}}\mathbf{F}d\mathbf{X}$$

$$\therefore ds^2 = d\mathbf{X}\cdot\mathbf{C}d\mathbf{X}$$

where $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, is called the *right Cauchy-Green deformation tensor*.

$$ds^2 = d\mathbf{X}\cdot\mathbf{C}d\mathbf{X}$$

We note that if **C=I**, then

$$ds^2 = d\mathbf{X}\cdot d\mathbf{X}$$

$$\Rightarrow ds^2 = dS^2.$$

Therefore, **C=I** corresponds to a rigid body motion (translation and/or rotation).
Now, we have

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}$$

$$\begin{aligned}\therefore \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} &= (\mathbf{I} + \nabla\mathbf{u})^{\mathrm{T}}(\mathbf{I} + \nabla\mathbf{u}) \\ &= (\mathbf{I} + (\nabla\mathbf{u})^{\mathrm{T}})(\mathbf{I} + \nabla\mathbf{u}) \\ &= \mathbf{I} + \boxed{\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}} + (\nabla\mathbf{u})^{\mathrm{T}}(\nabla\mathbf{u})}\end{aligned}$$

Let $\mathbf{E}^{*} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}} + (\nabla\mathbf{u})^{\mathrm{T}}(\nabla\mathbf{u})]$

then, $\mathbf{C} = \mathbf{I} + 2\mathbf{E}^{*}$

shows that the tensor $\mathbf{E}^{*}$ characterizes the changes of lengths in the continuum due to displacements of the material points. This tensor $\mathbf{E}^{*}$ is known as the *Lagrange strain tensor*. It is a finite deformation tensor.

In this section, we consider only cases where the components of the displacement vector as well as their partial derivatives are all very small (mathematically infinitesimal) so that the absolute value of every component of $(\nabla \mathbf{u})^{\mathrm{T}}(\nabla \mathbf{u})$ is a small quantity of higher order than those of the components of $(\nabla \mathbf{u})$. For such cases

$$\mathbf{C} \approx \mathbf{I} + 2\mathbf{E}$$

where

$$\mathbf{E} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}] = \text{Symmetric part of } (\nabla \mathbf{u})$$

This tensor **E** is known as *the infinitesimal strain tensor*. In Cartesian coordinates

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

In matrix form

$$[\mathbf{E}] = \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2} + \dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3} + \dfrac{\partial u_3}{\partial X_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_1} + \dfrac{\partial u_1}{\partial X_2}\right) & \dfrac{\partial u_2}{\partial X_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3} + \dfrac{\partial u_3}{\partial X_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial X_1} + \dfrac{\partial u_1}{\partial X_3}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial X_2} + \dfrac{\partial u_2}{\partial X_3}\right) & \dfrac{\partial u_3}{\partial X_3} \end{bmatrix}$$

Note:

Consider two material elements $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$. Due to motion, they become $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ at time t.

We have, for small deformation, from $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ and $\mathbf{C} \approx \mathbf{I} + 2\mathbf{E}$,

$$\begin{aligned} d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} &= \mathbf{F}d\mathbf{X}^{(1)} \cdot \mathbf{F}d\mathbf{X}^{(2)} = \mathbf{F}d\mathbf{X}^{(2)} \cdot \mathbf{F}d\mathbf{X}^{(1)} \\ &= d\mathbf{X}^{(1)} \cdot \mathbf{F}^{\mathrm{T}}\mathbf{F}d\mathbf{X}^{(2)} = d\mathbf{X}^{(1)} \cdot \mathbf{C}d\mathbf{X}^{(2)} \\ &= d\mathbf{X}^{(1)} \cdot (\mathbf{I} + 2\mathbf{E})d\mathbf{X}^{(2)} \end{aligned}$$

that is,

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} + 2d\mathbf{X}^{(1)} \cdot \mathbf{E}d\mathbf{X}^{(2)}$$

This equation will be used in the next section to establish the meaning of the components of the infinitesimal strain tensor **E**.

GEOMETRICAL MEANING OF THE COMPONENTS OF THE INFINITESIMAL STRAIN TENSOR

(a) *Diagonal elements* of **E**.

Consider the single material element $d\mathbf{X}^{(1)}=d\mathbf{X}^{(2)}=d\mathbf{X}=dS\mathbf{n}$, where **n** is a unit vector and dS is the length of $d\mathbf{X}$. Due to motion, $d\mathbf{X}$ becomes $d\mathbf{x}$ with a length of ds. Then $d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} + 2d\mathbf{X}^{(1)} \cdot \mathbf{E}d\mathbf{X}^{(2)}$ gives

$$d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X} \cdot d\mathbf{X} + 2dS\mathbf{n} \cdot \mathbf{E}dS\mathbf{n}$$

That is,

$$ds^2 = dS^2 + 2dS^2(\mathbf{n} \cdot \mathbf{E}\mathbf{n}) \tag{1}$$

For small deformation,

$$ds^2 - dS^2 = (ds - dS)(ds + dS) \approx 2dS(ds - dS)$$

Thus, Eq. (1) gives

$$\frac{ds-dS}{dS} = \mathbf{n} \cdot \mathbf{E}\mathbf{n} = E_{nn} \text{ (no sum on } n)$$

$$\frac{ds-dS}{dS} = \mathbf{n} \cdot \mathbf{E}\mathbf{n} = E_{nn} \text{ (no sum on n)}$$

This equation states that the unit elongation (i.e., increase in length per unit original length) for the element that was in the direction **n**, is given by $\mathbf{n} \cdot \mathbf{E}\mathbf{n}$. In particular, if the element was in the $\mathbf{e}_1$ direction in the reference state, then $\mathbf{n}=\mathbf{e}_1$ and $\mathbf{e}_1 \cdot \mathbf{E}\mathbf{e}_1 = E_{11}$, etc. Thus,

E_{11} is the unit elongation for an element originally in the x_1 direction.
E_{22} is the unit elongation for an element originally in the x_2 direction.
E_{33} is the unit elongation for an element originally in the x_3 direction.

These components (the diagonal elements of **E**) are also known as the *normal strains*.

(b) *The off diagonal elements* of **E**.

Let $d\mathbf{X}^{(1)}=dS_1\mathbf{m}$ and $d\mathbf{X}^{(2)}=dS_2\mathbf{n}$, where **m** and **n** are unit vectors perpendicular to each other. Due to motion, $d\mathbf{X}^{(1)}$ becomes $d\mathbf{x}^{(1)}$ with length ds_1 and $d\mathbf{X}^{(2)}$ becomes $d\mathbf{x}^{(2)}$ with length ds_2.

Let the angle between the two deformed vectors $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ be denoted by θ.

Since

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} + 2d\mathbf{X}^{(1)} \cdot \mathbf{E}d\mathbf{X}^{(2)}$$

then,

$$ds_1 ds_2 cos\theta = dS_1\mathbf{m} \cdot dS_2\mathbf{n} + 2dS_1\mathbf{m} \cdot \mathbf{E}dS_2\mathbf{n}$$

$$ds_1 ds_2 cos\theta = dS_1 dS_2(\mathbf{m} \cdot \mathbf{n}) + 2dS_1 dS_2(\mathbf{m} \cdot \mathbf{En})$$

$$ds_1 ds_2 cos\theta = 2dS_1 dS_2(\mathbf{m} \cdot \mathbf{En})$$

$$cos\theta = \frac{d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}}{|d\mathbf{x}^{(1)}||d\mathbf{x}^{(2)}|}$$

$$cos\theta = \frac{d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}}{ds_1 ds_2}$$

If we let

$$\theta = \frac{\pi}{2} - \gamma$$

then γ measures the small decrease in angle between $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ (known as the *shear strain*) due to deformation. Since

$$cos\left(\frac{\pi}{2} - \gamma\right) = sin\gamma$$

and for small strain

$$sin\gamma \approx \gamma, \frac{ds_1}{dS_1} \approx 1, \frac{ds_2}{dS_2} \approx 1$$

therefore $ds_1 ds_2 cos\theta = 2dS_1 dS_2(\mathbf{m} \cdot \mathbf{En})$ becomes

$$\frac{ds_1}{dS_1}\frac{ds_2}{dS_2} cos\theta = 2\,(\mathbf{m} \cdot \mathbf{En})$$

$$\gamma = 2\,(\mathbf{m} \cdot \mathbf{En})$$

$$\gamma = 2\,(\mathbf{m} \cdot \mathbf{En})$$

In particular, if the elements were in the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions before deformation, then $\mathbf{m} \cdot \mathbf{En} = \mathbf{e}_1 \cdot \mathbf{Ee}_2 = E_{12}$, etc., so that, according to Equation above:

$2E_{12}$ gives the decrease in angle between two elements initially in the x_1 and x_2 directions.

$2E_{13}$ gives the decrease in angle between two elements initially in the x_1 and x_3 directions.

$2E_{23}$ gives the decrease in angle between two elements initially in the x_2 and x_3 directions.

Example

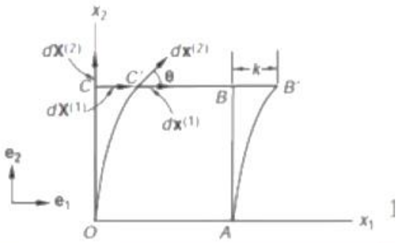
Given the following displacement components

$$u_1 = kX_2^2,\ u_2 = u_3 = 0. \quad k = 10^{-4}$$

(a) Obtain the infinitesimal strain tensor **E**.

(b) Using the strain tensor **E**, find the unit elongation for the material elements $d\mathbf{X}^{(1)}=d\mathbf{X}_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)}=d\mathbf{X}_2\mathbf{e}_2$, which were at the point $C(0,1,0)$ of figure below. Also find the decrease in angle between these two elements.

(c) Compare the results with those of the same Example in the last lecture.



(a) Obtain the infinitesimal strain tensor **E**.

Solution:

We have

$$[\nabla \mathbf{u}] = \begin{bmatrix} 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

therefore,

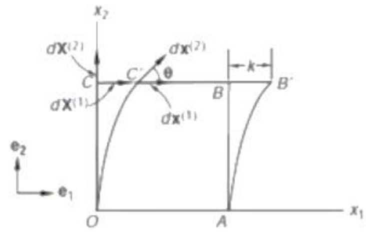
$$[\mathbf{E}] = [(\nabla \mathbf{u})^S] = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$$

$$[\mathbf{E}] = \frac{1}{2}\left(\begin{bmatrix} 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2kX_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & kX_2 & 0 \\ kX_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Using the strain tensor **E**, find the unit elongation for the material elements $d\mathbf{X}^{(1)}=d\mathbf{X}_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)}=d\mathbf{X}_2\mathbf{e}_2$, which were at the point $C(0,1,0)$ of figure below. Also find the decrease in angle between these two elements.

Solution:

At point C, $X_2=1$, therefore



$$[\mathbf{E}] = \begin{bmatrix} 0 & kX_2 & 0 \\ kX_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

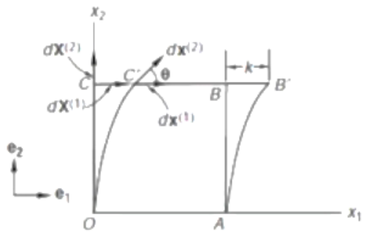
For the element $d\mathbf{X}^{(1)}=d\mathbf{X}_1\mathbf{e}_1$, the unit elongation is E_{11}, which is zero.
For the element $d\mathbf{X}^{(2)}=d\mathbf{X}_2\mathbf{e}_2$, the unit elongation is E_{22}, which is also zero.
The decrease in angle between these elements is given by $2E_{12}$, which is equal to $2k$, i.e., 2×10^{-4} radians.

(c) Compare the results with those of the same Example in the last lecture.

Solution:

In the last lecture, we found that

$\frac{|d\mathbf{x}^{(1)}|}{|d\mathbf{X}^{(1)}|}=1$ and $\frac{|d\mathbf{x}^{(2)}|}{|d\mathbf{X}^{(2)}|}=\sqrt{4k^2+1}$ and $sin\gamma = 2k$



i.e.

$$\frac{|d\mathbf{x}^{(1)}|-|d\mathbf{X}^{(1)}|}{|d\mathbf{X}^{(1)}|}=0 \text{ and } \frac{|d\mathbf{x}^{(2)}|-|d\mathbf{X}^{(2)}|}{|d\mathbf{X}^{(2)}|}=\sqrt{4k^2+1}\text{-}1$$

$$\approx 1+2k^2-1 \approx 2k^2 \approx 0$$

and

$$\gamma \approx 2 \times 10^{-4}$$

Example

Given the displacement field

$$u_1 = k(2X_1 + X_2^2), \quad u_2 = k(X_1^2 - X_2^2), \quad u_3 = 0. \quad k = 10^{-4}$$

(a) Find the unit elongation and the change of angle for the two material elements $d\mathbf{X}^{(1)}=d\mathbf{X}_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)}=d\mathbf{X}_2\mathbf{e}_2$ that emanate from a particle designated by $X = \mathbf{e}_1 - \mathbf{e}_2$.

(b) Find the deformed position of these two elements: $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$.

(a) Find the unit elongation and the change of angle for the two material elements $d\mathbf{X}^{(1)}=d\mathbf{X}_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)}=d\mathbf{X}_2\mathbf{e}_2$ that emanate from a particle designated by $X = \mathbf{e}_1 - \mathbf{e}_2$.

Solution:

We evaluate $[\nabla\mathbf{u}]$ and $[\mathbf{E}]$ at $(X_1, X_2, X_3) = (1, -1, 0)$ as

$$[\nabla\mathbf{u}]_{(1,-1,0)} = \begin{bmatrix} 2k & 2kX_2 & 0 \\ 2kX_1 & -2kX_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(1,-1,0)} = \begin{bmatrix} 2k & -2k & 0 \\ 2k & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

therefore, $[\mathbf{E}] = [(\nabla\mathbf{u})^{\mathbf{S}}] = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{T}})$

$$[\mathbf{E}] = \frac{1}{2}\left(\begin{bmatrix} 2k & -2k & 0 \\ 2k & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2k & 2k & 0 \\ -2k & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{E}] = \begin{bmatrix} 2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $E_{11} = E_{22} = 2k$, both elements have a unit elongation of 2×10^{-4}. Further, since $E_{12} = 0$, these line elements remain perpendicular to each other.

(b) We know that $d\mathbf{x} = d\mathbf{X} + (\nabla\mathbf{u})d\mathbf{X}$, so

$$[d\mathbf{x}^{(1)}] = [d\mathbf{X}^{(1)}] + [\nabla\mathbf{u}][d\mathbf{X}^{(1)}]$$

$$[d\mathbf{x}^{(1)}] = \begin{bmatrix} dX_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2k & -2k & 0 \\ 2k & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} dX_1 \\ 0 \\ 0 \end{bmatrix}$$

$$[d\mathbf{x}^{(1)}] = \begin{bmatrix} dX_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2kdX_1 \\ 2kdX_1 \\ 0 \end{bmatrix} = \begin{bmatrix} dX_1 + 2kdX_1 \\ 2kdX_1 \\ 0 \end{bmatrix} = dX_1\begin{bmatrix} 1 + 2k \\ 2k \\ 0 \end{bmatrix}$$

$$[d\mathbf{x}^{(2)}] = [d\mathbf{X}^{(2)}] + [\nabla\mathbf{u}][d\mathbf{X}^{(2)}]$$

$$[d\mathbf{x}^{(2)}] = \begin{bmatrix} 0 \\ dX_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2k & -2k & 0 \\ 2k & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ dX_2 \\ 0 \end{bmatrix}$$

$$[d\mathbf{x}^{(2)}] = \begin{bmatrix} 0 \\ dX_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2kdX_2 \\ 2kdX_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2kdX_2 \\ dX_2 + 2kdX_2 \\ 0 \end{bmatrix} = dX_2 \begin{bmatrix} -2k \\ 1+2k \\ 0 \end{bmatrix}$$

$$[d\mathbf{X}^{(1)}] \rightarrow [d\mathbf{x}^{(1)}] \qquad \begin{bmatrix} dX_1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} dX_1 + 2kdX_1 \\ 2kdX_1 \\ 0 \end{bmatrix}$$

$$[d\mathbf{X}^{(2)}] \rightarrow [d\mathbf{x}^{(2)}] \qquad \begin{bmatrix} 0 \\ dX_2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2kdX_2 \\ dX_2 + 2kdX_2 \\ 0 \end{bmatrix}$$

The deformed position of these elements are sketched in the following figure. Note from the diagram that

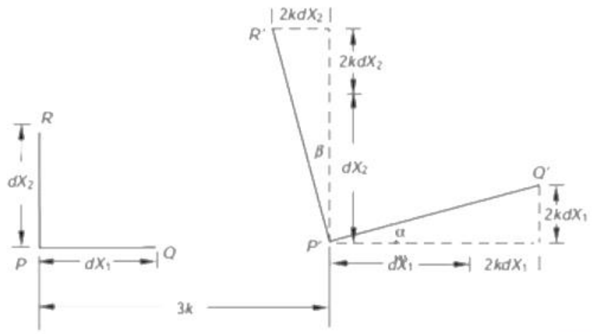
$$\alpha \approx tan\alpha = \frac{2kdX_1}{dX_1(1+2k)} = \frac{2k}{1+2k} \approx 2k$$

and

$$\beta \approx tan\beta = \frac{2kdX_2}{dX_2(1+2k)} = \frac{2k}{1+2k} \approx 2k$$

Thus, as previously obtained, there is no change of angle between $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$

.



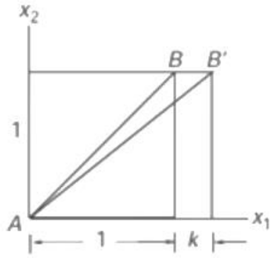
Example

A unit cube with edges parallel to the coordinate axes is given a displacement field

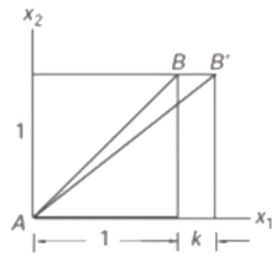
$$u_1 = kX_1, \quad u_2 = u_3 = 0. \qquad k = 10^{-4}$$

Find the increase in length of the diagonal AB (see the figure below)

(a) by using the infinitesimal strain tensor E and **(b)** by geometry.



Solution: (a) We have $u_1 = kX_1, \quad u_2 = u_3 = 0$. Then



$$\nabla\mathbf{u} = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [\mathbf{E}] = [(\nabla\mathbf{u})^S] = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$$

$$[\mathbf{E}] = \frac{1}{2}\left(\begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the diagonal element was originally in the direction $\mathbf{n} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$, its unit elongation is given by

$$E_{nn} = \mathbf{n}\cdot\mathbf{En} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{k}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} = \frac{k}{2}$$

Since $AB = \sqrt{2} \Rightarrow \Delta AB = \frac{k}{2}\sqrt{2} = \frac{k}{\sqrt{2}}$

(b) Geometrically,

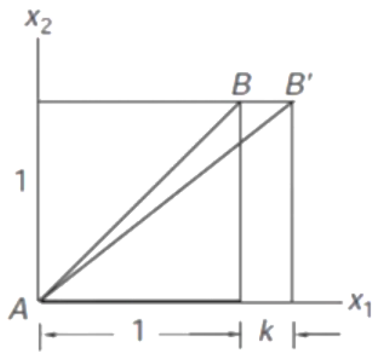
$$\Delta AB = AB' - AB$$

$$\Delta AB = [1 + (1 + k)^2]^{\frac{1}{2}} - \sqrt{2}$$

$$\Delta AB = [1 + 1 + 2k + k^2]^{\frac{1}{2}} - \sqrt{2}$$

$$\Delta AB = [2 + 2k + k^2]^{\frac{1}{2}} - \sqrt{2}$$

$$\Delta AB = \sqrt{2}\left[1 + k + \frac{k^2}{2}\right]^{\frac{1}{2}} - \sqrt{2} \qquad (1)$$



Now,

$$\left[1 + k + \frac{k^2}{2}\right]^{\frac{1}{2}} = 1 + \frac{1}{2}\left(k + \frac{k^2}{2}\right) + \cdots \approx 1 + \frac{k}{2} \qquad (2)$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$(1 + x)^n \approx 1 + nx$$

Put (2) in (1)

$$\Delta AB = \sqrt{2}\left[1 + \frac{k}{2}\right] - \sqrt{2} = \sqrt{2} + \frac{k}{\sqrt{2}} - \sqrt{2} = \frac{k}{\sqrt{2}}$$

which is the same result of part (a)